Orbits of G on $V \times G/B \times G/B$ in type A
Orbits of K on $V \times G/P$ in type A

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Part I – Orbits of G on $V \times G/B \times G/B$ in type A

- $V = \mathbf{C}^n$, n-dimensional complex vector space
- $G = GL_n(\mathbf{C})$, the general linear group (invertible matrices)
- $(e_1, e_2, \ldots, e_n)$ is a fixed basis of V
- Notation for permutations: w sends 123… to $w_1 w_2 w_3 \ldots$

Definitions

- Flag : $\{0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \ldots \subset \langle v_1, \ldots, v_k \rangle \subset V\}$
- B = Borel subgroup of G, consists of invertible upper triangular matrices, the stabilizer of the complete flag $\{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_n \rangle\}$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix}$$

- G/B = complete flag variety $\{0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V\}$

Preliminary: Orbits of B on G/B (Bruhat decomposition)

- Orbit representatives: F_w (w is a permutation)
  - $(e_1, \ldots, e_n)$ is a fixed basis of V
  - $F_w = \{0 \subset \langle e_{w\_1} \rangle \subset \langle e_{w\_1}, e_{w\_2} \rangle \subset \ldots \subset V\}$
    - e.g. $\{\langle e_2 \rangle \subset \langle e_2, e_1 \rangle \subset \langle e_2, e_1, e_3 \rangle\}$ represents $n = 3$, w: 123 → 213
- Each F_w lies in a different orbit
- Each $F \in G/B$ is in the same B-orbit as some F_w

Preliminary: Orbits of G on $G/B \times G/B$

- Same as orbits of B on G/B (group-theoretic result)
- Orbit representatives: (F_e, F_w)
  - $F_w = \{0 \subset \langle e_{w\_1} \rangle \subset \langle e_{w\_1}, e_{w\_2} \rangle \subset \ldots \subset V\}$
  - $F_e = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_n \rangle\}$

Orbits of G on V × G/B × G/B

Proposition 1. Each of the following elements of $V \times G/B \times G/B$ is a representative of a unique orbit of G on $V \times G/B \times G/B$, and together they represent all such orbits: (v, F_e, F_w), where w spans all permutations on n letters and v represents any $\sum_{s \in s_b} e_s$ such that $s_b \subseteq \{1, 2, ..., n\}$ and if $w_i, w_j \in s_b$, then $i < j \leftrightarrow w_i > w_j$ (i.e. the elements of s_b appear in descending order in $w_1 w_2 w_3 ...$).

Orbits of G on V × G/B × G/B

- Any $(v, F_1, F_2) \in V \times G/B \times G/B$ is in the same orbit as some (v', F_e, F_w)
- The orbits of G_w on V are the orbits of G on $V \times G/B \times G/B$
- n = 4; let w: 1234 → 1342, and let G_w stabilize (F_e, F_w).

$$F_e = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset V\}$$
$$F_w = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_3, e_4 \rangle \subset V\}$$

$$(F_e, F_w) \text{ stabilizer} : \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orbits of G on V × G/B × G/B

- Example: n = 4, w can be any permutation
  - If w: 1234 → 1342, then possible s_b are ∅, {1}, {2}, {3}, {4}, {3,2}, {4,2} (s_b must be descending subset of $w_1w_2w_3w_4$)

s_b	v	Orbit
$\varnothing$	0	0
$\{1\}$	e_1	$\langle e_1\rangle\backslash 0$
$\{2\}$	e_2	$\langle e_1, e_2\rangle\backslash\langle e_1\rangle$
$\{3\}$	e_3	$\langle e_1, e_3\rangle\backslash\langle e_1\rangle$
$\{4\}$	e_4	$\langle e_1, e_3, e_4\rangle\backslash\langle e_1, e_3\rangle$
$\{2,3\}$	e_2+e_3	$\langle e_1, e_2, e_3\rangle\backslash(\langle e_1, e_2\rangle \cup \langle e_1, e_3\rangle)$
$\{2,4\}$	e_2+e_4	$\langle e_1, e_2, e_3, e_4\rangle\backslash(\langle e_1, e_2, e_3\rangle \cup \langle e_1, e_3, e_4\rangle)$

Part II – Orbits of K on G/P in type A

- $K = Sp_{2n}(\mathbf{C})$, the symplectic group
- P = a parabolic subgroup of G (stabilizer of a partial flag)
  - G/P = a partial flag variety, $\{0 \subset V_{i\_1} \subset V_{i\_2} \subset \ldots \subset V_{i\_m} \subset V\}$

Orbits of K on $V \times G/P$

- Symplectic form ($V \times V \to \mathbf{C}$)
  - $\omega(v, w) = v_1w_4 + v_2w_3 - v_3w_2 - v_4w_1$ (in $2n = 4$)
- Properties:
  - Bilinear: $\omega(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1\, \omega(v_1, w) + \lambda_2\, \omega(v_2, w)$
    - $\omega(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1\, \omega(v, w_1) + \lambda_2\, \omega(v, w_2)$
  - $\omega(v, w) = -\omega(w, v)$ (skew-symmetric)
  - $\omega(v, v) = 0$ (totally isotropic)
  - If $\omega(v, w) = 0$ for all w, then $v = 0$ (nondegenerate)

Orbits of K on $V \times G/P$

- Perpendicularity: $v \perp w$ iff $\omega(v, w) = 0$
- For the unit vectors, e_i not perp. to $e_j \leftrightarrow i + j = 2n+1$
  - $\omega(e_1, e_4) = 1$, $\omega(e_2, e_3) = 1$
  - $\omega(e_i, e_j) = 0$ for all other $i \leq j$

Orbits of K on $V \times G/P$

- Symplectic matrix: preserves the symplectic form
  - $\omega(k.v, k.w) = \omega(v, w)$
  - A matrix k is symplectic iff $\omega(k_i, k_j) = \omega(e_i, e_j)$ for all (i, j) (where k_i is the i^{th} column of k)

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 6 & -1 \\ 1 & 7 & 11 & -2 \\ 3 & 0 & -1 & 1 \end{pmatrix}$$

Orbits of K on $V \times G/P$

- Technique
  - Same idea as orbits of G on $V \times G/B \times G/B$
    - Take a representative of each orbit of G/P
    - Find the stabilizer G_s in G
    - $K_s = G_s \cap K$
    - Determine orbits of K_s on V
- Example:
  - $G/P = \{0 \subset V_1 \subset V_2 \subset V\}$ with $2n = 4$

Orbits of K on $V \times G/P$ where $G/P = \{0 \subset V_1 \subset V_2 \subset V\}$

- $G/P = \{0 \subset <v_1> \subset <v_1, v_2> \subset V\}$ and $2n = 4$
- Two orbits of K on G/P
  - v_1 perpendicular to v_2 – choose to stabilize $\{<e_1> \subset <e_1, e_2>\}$
  - v_1 not perpendicular to v_2 – choose to stabilize $\{<e_1> \subset <e_1, e_4>\}$

Orbits of K on $V \times G/P$ where $G/P = \{0 \subset V_1 \subset V_2 \subset V\}$

- Perpendicular case – stabilizing $\{\langle e_1\rangle \subset \langle e_1, e_2\rangle\}$

$$G_s = \begin{bmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \qquad K_s = G_s \cap K$$

- 0 is its own orbit
- $K_1.e_1 = \langle e_1\rangle \backslash 0$
- $K_1.e_2 = \langle e_1, e_2\rangle \backslash \langle e_1\rangle$
- $K_1.e_3 = \langle e_1, e_2, e_3\rangle \backslash \langle e_1, e_2\rangle$
- $K_1.e_4 = \langle e_1, e_2, e_3, e_4\rangle \backslash \langle e_1, e_2, e_3\rangle$

Orbits of K on V × G/P where G/P = {0 ⊂ V_1 ⊂ V_2 ⊂ V }

- Nonperpendicular case – stabilizing {<e_1> ⊂ <e_1, e_4>}

$$G_s = \begin{bmatrix} 1 & * & * & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & * & * & * \end{bmatrix} \qquad K_s = G_s \cap K$$

- 0 is its own orbit
- $K_2.e_1 = \langle e_1 \rangle \backslash 0$
- $K_2.e_4 = \langle e_1, e_4 \rangle \backslash \langle e_1 \rangle$
- $K_2.e_2 = \langle e_2, e_3 \rangle \backslash 0$
- $K_2.(e_1 + e_2) = \langle e_1, e_2, e_3 \rangle \backslash (\langle e_1 \rangle \cup \langle e_2, e_3 \rangle)$
- $K_2.(e_2 + e_4) = \langle e_1, e_2, e_3, e_4 \rangle \backslash (\langle e_1, e_4 \rangle \cup \langle e_1, e_2, e_3 \rangle)$

Orbits of K on $V \times G/P$ where $G/P = \{0 \subset V_1 \subset V\}$

- $K = Sp_{2n}$ acts on G/P transitively (with one orbit)
- K_s is stabilizer of a point in G/P
- Orbits of K_s on V:
  - 0
  - $\langle e_1 \rangle \backslash 0$
  - $\langle e_1, ..., e_{2n-1} \rangle \backslash \langle e_1 \rangle$
  - $\langle e_1, ..., e_{2n} \rangle \backslash \langle e_1, ..., e_{2n-1} \rangle$

Orbits of K on V × G/P where G/P = {0 ⊂ V_2 ⊂ V }

- Two orbits of K on G/P
  - Perpendicular case: stabilizing $\langle e_1, e_2 \rangle$
  - Nonperpendicular case: stabilizing $\langle e_1, e_{2n} \rangle$
- Orbits of K_s on V in each case:

Perpendicular case:

- 0
- $\langle e_1, e_2\rangle \backslash 0$
- $\langle e_1, ..., e_{2n-2}\rangle \backslash \langle e_1, e_2\rangle$
- $\langle e_1, ..., e_{2n}\rangle \backslash \langle e_1, ..., e_{2n-2}\rangle$

Nonperpendicular case:

- 0
- $\langle e_1, e_{2n}\rangle \backslash 0$
- $\langle e_2, ..., e_{2n-1}\rangle \backslash 0$
- $\langle e_1, ..., e_{2n}\rangle \backslash (\langle e_1, e_{2n}\rangle \cup \langle e_2, ..., e_{2n-1}\rangle)$

Orbits of K on $V \times G/P$

- General problem: K-orbits on $V \times G/P$ for general G/P
  - $K = Sp_{2n}(\mathbf{C})$
  - $G/P = \{0 \subset V_{i\_1} \subset V_{i\_2} \subset \ldots \subset V_{i\_k} \subset V\}$

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