# A New Approach to $q$-Enumeration of Modular Statistics

William Kuszmaul

*Third Annual PRIMES Conference*

Mentored by Darij Grinberg

May 18, 2013

# Example of enumerating a modular statistic

Consider integers $x$ from 0 to 7.

$f : \mathbb{Z} \to \mathbb{Z}$

$f(x) \bmod n$

| $x$ | $x^2$ | $x^2 \bmod 5$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 2 | 4 | 4 |
| 3 | 9 | 4 |
| 4 | 16 | 1 |
| 5 | 25 | 0 |
| 6 | 36 | 1 |
| 7 | 49 | 4 |

Question: What is

$|\{x : x^2 \equiv 4 \pmod 5\}|$?

Answer: 3

A modular statistic counts (enumerates) the # of rows that share a given modular answer.

# A general problem

**The question:** Let $M$ be a finite set and $f : M \to \mathbb{Z}$. How many $a \in M$ have $f(a) \equiv i \bmod n$ for a given $i$ and $n$?

- $M$ can contain anything: paths, words, numbers, etc.
- Two variables to remember: $i$ and $n$.

**A step forward:** I find a restructuring of the problem that often yields a simple solution.

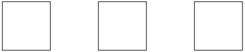
# A definition: Nontrivially periodic vector

**Definition**

*A **nontrivially periodic vector** of length $n$ repeats every $k$ positions for some $k|n$ where $k < n$.*

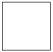
Examples:

$\langle 1, 1, 1, 1, 1, 1 \rangle$



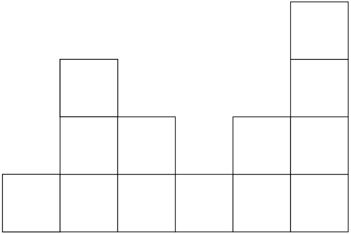
$\langle 0, 1, 0, 1, 0, 1 \rangle$

*Not* nontrivially periodic:

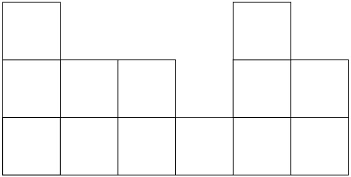


$\langle 1, 0, 0, 0, 0, 0 \rangle$

# Playing a game



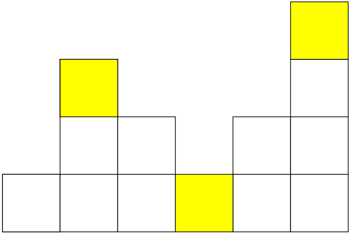
$\langle 1, 3, 2, 1, 2, 4 \rangle$



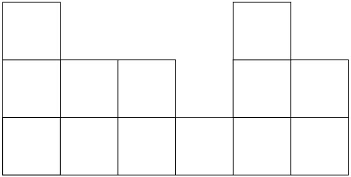
$\langle 3, 2, 2, 1, 3, 2 \rangle$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



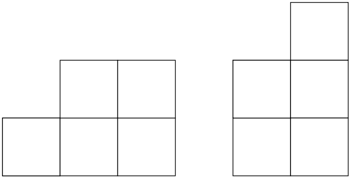
$$\begin{array}{r} \langle 1,3,2,1,2,4\rangle \\ -\langle 0,1,0,1,0,1\rangle \\ \hline \langle 1,2,2,0,2,3\rangle \end{array}$$



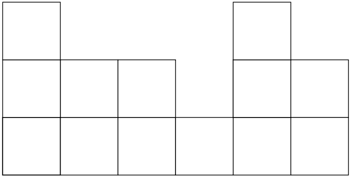
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



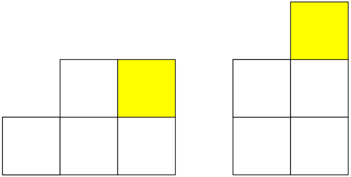
$$\begin{array}{r} \langle 1,3,2,1,2,4\rangle \\ -\langle 0,1,0,1,0,1\rangle \\ \hline \langle 1,2,2,0,2,3\rangle \end{array}$$



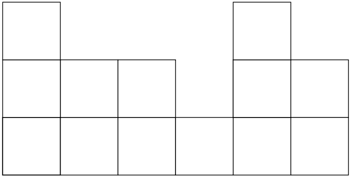
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



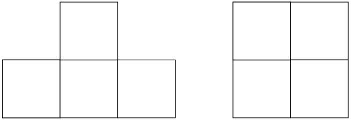
$$\begin{array}{r} \langle 1,2,2,0,2,3\rangle \\ -\langle 0,0,1,0,0,1\rangle \\ \hline \langle 1,2,1,0,2,2\rangle \end{array}$$



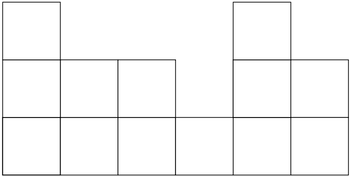
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



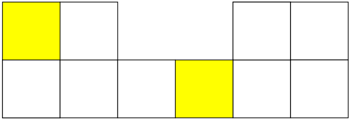
$$\begin{array}{r} \langle 1,2,2,0,2,3\rangle \\ -\langle 0,0,1,0,0,1\rangle \\ \hline \langle 1,2,1,0,2,2\rangle \end{array}$$



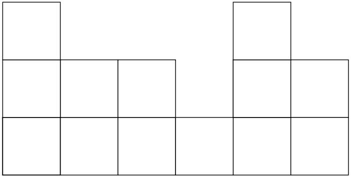
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



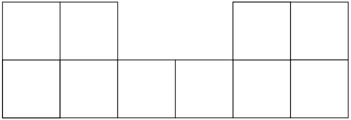
$$\begin{array}{r} \langle 1,2,1,0,2,2\rangle \\ +\langle 1,0,0,1,0,0\rangle \\ \hline \langle 2,2,1,1,2,2\rangle \end{array}$$



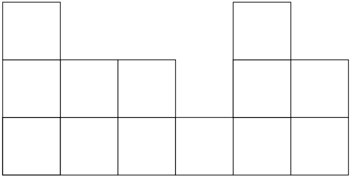
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



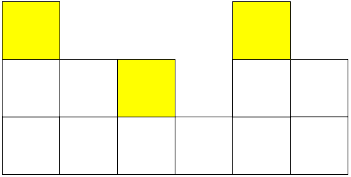
$$\begin{array}{r} \langle 1,2,1,0,2,2\rangle \\ +\langle 1,0,0,1,0,0\rangle \\ \hline \langle 2,2,1,1,2,2\rangle \end{array}$$



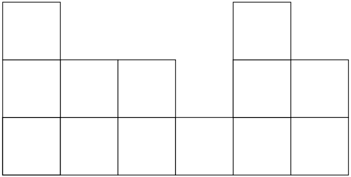
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



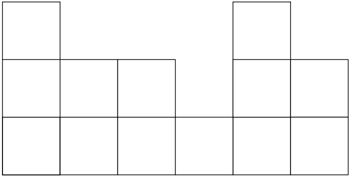
$$\begin{array}{r} \langle 2,2,1,1,2,2\rangle \\ +\langle 1,0,1,0,1,0\rangle \\ \hline \langle 3,2,2,1,3,2\rangle \end{array}$$



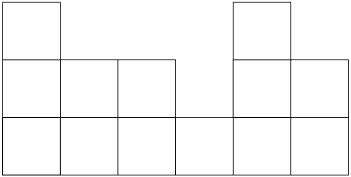
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



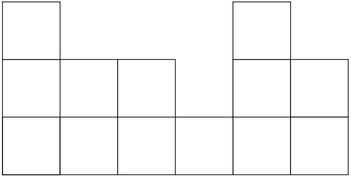
$$\begin{array}{r} \langle 2,2,1,1,2,2\rangle \\ +\langle 1,0,1,0,1,0\rangle \\ \hline \langle 3,2,2,1,3,2\rangle \end{array}$$



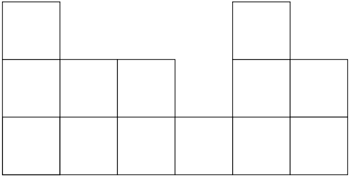
$$\langle 3,2,2,1,3,2\rangle$$

Goal: Connect two vectors by adding and subtracting nontrivially periodic vectors.

# Playing a game



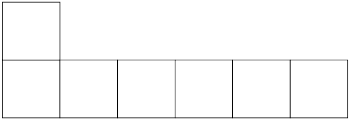
$\langle 3, 2, 2, 1, 3, 2 \rangle$



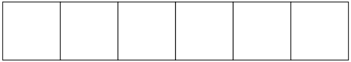
$\langle 3, 2, 2, 1, 3, 2 \rangle$

We win! If we can win the game using $\overrightarrow{a}$ and $\overrightarrow{b}$ as our vectors, we say that $\overrightarrow{a}$ and $\overrightarrow{b}$ **equivalent** under the period game equivalence.

# Sometimes we cannot win



$\langle 2, 1, 1, 1, 1, 1 \rangle$



$\langle 1, 1, 1, 1, 1, 1 \rangle$

No way to connect these two vectors.

# Function $G$ tells us when we can win

- $G$ maps vectors to vectors.

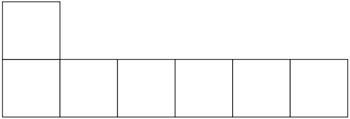
**Theorem**

$$G(\overrightarrow{a}) = G(\overrightarrow{b})$$

$$\Updownarrow$$

*We can win the game using $\overrightarrow{a}$ and $\overrightarrow{b}$ as our vectors.*

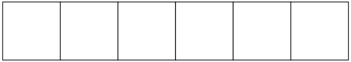
# An example where we cannot win



$$\langle 2,1,1,1,1,1\rangle$$

$$G(\langle 2,1,1,1,1,1\rangle)$$

$$= \langle 2,1,-1,-2,-1,1\rangle$$



$$\langle 1,1,1,1,1,1\rangle$$

$$G(\langle 1,1,1,1,1,1\rangle)$$

$$= \langle 0,0,0,0,0,0\rangle$$

No way to connect these two vectors.

# What is $G$ of $\overrightarrow{x}$ (a vector of size $n$)?

- Let $T_d^j(\overrightarrow{x}) = \sum_{k \equiv j \bmod d} \overrightarrow{x}_k$.
- Let $G_j(\overrightarrow{x}) = \sum_{d|n} d\, T_d^j\, \mu\left(\frac{n}{d}\right)$.
- Then $G(\overrightarrow{x}) = \langle G_0, G_1, \ldots, G_{n-1} \rangle$.

Why does it work?

- Proof of invariance is combinatorial.
- Proof of exhaustiveness related to norms of cyclotomic integers. Proved by my mentor Darij Grinberg.

# Restructuring the problem: A counting theorem

- Let $M$ be a finite set and $f : M \to \mathbb{Z}$. Pick $i$ and $n$.
- For each $d|n$, let $\overrightarrow{X(d)}$ be the vector with $\overrightarrow{X(d)}_j =$ number of $b \in M$ with $f(b) \equiv j \bmod d$.
- For $\overrightarrow{X(d)}$, pick **any** vector $\overrightarrow{A(d)}$ that is equivalent to $\overrightarrow{X(d)}$ under the period game equivalence.
  - $\overrightarrow{A(d)}$ can be picked to be much simpler than $\overrightarrow{X(d)}$.

**Theorem**

*The number of $a \in M$ with $f(a) \equiv i \bmod n$ is*

$$\frac{1}{n} \sum_{d|n} G_i(\overrightarrow{A(d)}).$$

# Some previously unsolved problems

**A new result:** The number of words, each with major index $\equiv i \bmod n$, consisting of the letters $1, 2, \ldots$ each appearing $a_1, a_2, \ldots$ times respectively where $n|(a_1 + a_2 + \cdots)$ is

$$\sum_{d|n,a_1,a_2,\ldots} \left( \frac{(\frac{a_1}{d} + \frac{a_2}{d} + \frac{a_3}{d} + \cdots)!}{\frac{a_1}{d}!\frac{a_2}{d}!\frac{a_3}{d}!\cdots} \sum_{k|d,i} \frac{k}{n}\mu(\frac{d}{k}) \right).$$

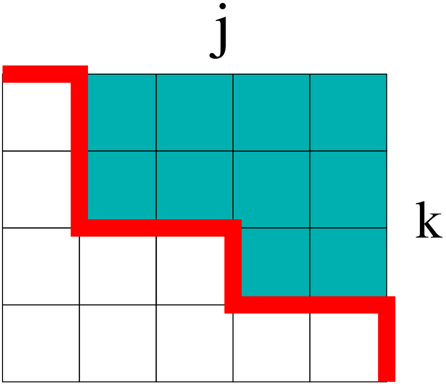
**Another new result:** The number of Catalan paths with major index $\equiv i \bmod n$ on a $j \times j$ grid with $n|2j$ is

$$C_j/n + \sum_{\substack{d|n,j \\ d\neq 1}} \left( \binom{2j/d}{j/d} \sum_{k|d,i} \frac{k}{n}\mu(\frac{d}{k}) \right).$$

# An application: Area of monotonic paths

- From top left of grid to bottom right of grid.
- Goes only right and down.

Question: Let $n|(j+k)$. How many paths on a $j \times k$ grid have area $\equiv i \bmod n$ for a given $i$ and $n$?




area=10
path on a $5 \times 4$ grid

(Previously solved by Reiner, Stanton, and White.)

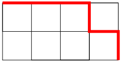
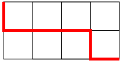
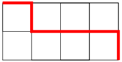
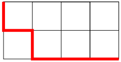
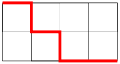
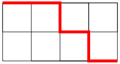
# A tool: cyclic shifts

- A path corresponds with a word of letters
  r (right) and d (down).
- The **cyclic shift** of a word is the same word, but with the last letter killed and inserted as the first letter.
  - e.g.

rrdrrddd
drrdrrdd
ddrrdrrd
dddrrdrr
rdddrrdr
rrdddrrd
drrdddrr
rdrrdddr

- The **cyclic shift** of a path is the path corresponding with the cyclic shift of its word.

# EXAMPLE ON A $4 \times 2$ GRID

| | | |
|---|---|---|
|  rrrdrd<br>area $\equiv 1 \bmod 6$ |  drrrdr<br>area $\equiv 5 \bmod 6$ |  rdrrrd<br>area $\equiv 3 \bmod 6$ |
|  drdrrr<br>area $\equiv 1 \bmod 6$ |  rdrdrr<br>area $\equiv 5 \bmod 6$ |  rrdrdr<br>area $\equiv 3 \bmod 6$ |

- Each cyclic shift either kills a column or adds a row.
- Each cyclic shift changes area by 4 mod 6.

# Finding a simple $\overrightarrow{A(6)}$ for a $4 \times 2$ grid

- Each cyclic shift changes area by 4 mod 6. Modulo 6, the areas $1, 5, 3, 1, 5, 3$ appear.
- The modular statistics of the areas modulo 6 of the resulting paths form a nontrivially periodic vector: $\langle 0, 2, 0, 2, 0, 2 \rangle$.
- $\implies$ we do not need to consider them in $\overrightarrow{A(6)}$.
- $\implies$ we can cancel out all paths in this way.

So we can simply pick

$$\overrightarrow{A(6)} = \langle 0, 0, 0, 0, 0, 0 \rangle.$$

## A simple proof of a known result

- Using cyclic shifts, we find that $\overrightarrow{A(d)}$ is $\langle 0,0,0,0,\ldots\rangle$ when not $d|j,k$ and $\langle \binom{(j+k)/d}{j/d}, 0, 0, \ldots\rangle$ when $d|j,k$.
- We finish the problem by plugging $\overrightarrow{A(d)}$ into

$$\frac{1}{n}\sum_{d|n} G_i(\overrightarrow{A(d)}).$$

**Theorem**

*The number of monotonic paths on a $j \times k$ grid with $n|(j+k)$ and with area $\equiv i \bmod n$ is*

$$\sum_{d|n,j}\left(\binom{(j+k)/d}{j/d}\sum_{r|d,i}\frac{r}{n}\mu(\frac{d}{r})\right).$$

# FUTURE WORK

- Study relations between our results and the cyclic sieving phenomenon.
- Find additional enumerative applications of our main result.
    - Let $\lambda$ be a partition of $n$. Can one prove *combinatorially* that the number of Standard Young Tableaux $T$ of shape $\lambda$ such that $T$ has major index $\equiv i \mod n$ depends only on $\lambda$ and the $\gcd(i, n)$?
- To continue studying the period game equivalence.

# Acknowledgements

I want to thank

1. My mentor Darij Grinberg.
2. MIT Professor Stanley for his encouragement.
3. MIT PRIMES for giving me the opportunity to conduct this research.

# Related work


1. J. Haglund. *The $q, t$-Catalan numbers and the space of diagonal harmonics: with an appendix on the combinatorics of Macdonald polynomials*, volume 41. American Mathematical Soc., 2008.
2. V. Reiner, D. Stanton, and D. White. *The cyclic sieving phenomenon*. Journal of Combinatorial Theory, Series A, 108(1):1750, 2004.
3. B.E. Sagan. *The cyclic sieving phenomenon: a survey*. arXiv preprint arXiv:1008.0790, 2010.
4. R. Stanley, *Enumerative Combinatorics Volume 1*, no. 49 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.