Lower bounds for the Crossing Number of the Cartesian Product of a Vertex-transitive Graph with a Cycle

Junho Won
MIT-PRIMES

December 24, 2013

Abstract. The minimum number of crossings for all drawings of a given graph G on a plane is called its crossing number, denoted $cr(G)$. Exact crossing numbers are known only for a few families of graphs, and even the crossing number of a complete graph K_m is not known for all m. Wenping et al. showed that $cr(K_m \Box C_n) \geqslant n \cdot cr(K_{m+2})$ for $n \geqslant 4$ and $m \geqslant 4$. We adopt their method to find a lower bound for $cr(G \Box C_n)$ where G is a vertex-transitive graph of degree at least 3. We also suggest some particular vertex-transitive graphs of interest, and give two corollaries that give lower bounds for $cr(G \Box C_n)$ in terms of n, $cr(G)$, the number of vertices of G, and the degree of G, which improve on Wenping et al.'s result.

1 Introduction

For basic definitions and notations that are not explained, the readers are referred to Diestel [3]. If G is a graph, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. C_n, or the n-cycle, is the graph with some n vertices $\{v_1, ..., v_n\}$ with an edge set $\{v_1v_2, ..., v_{n-1}v_n, v_nv_1\}$. K_n, or the complete graph on n vertices, is the simple graph with n vertices in which any two vertices are joined by an edge. $K_{l,m}$ denotes the graph whose vertex set can be partitioned into two subsets of size l and m, such that any two vertices in the same subset are not joined and any two vertices in different subsets are joined.

Calculating the crossing number of a given graph is a major area of research in topological graph theory. It has proven to be a very difficult task, and there are only few families of graphs whose crossing numbers are known. In fact, in 1983 Garey and Johnson [3] showed that the calculation is NP-complete. However, crossing numbers of some graphs are known, and one of the most interesting families of graphs have been the Cartesian products of two elementary graphs, such as paths, cycles, stars, complete graphs, complete bipartite or multipartite graphs (see, for example, Klešč [5]). A major result was achieved by Glebsky and Salazar [4] when they calculated the crossing number of the Cartesian product of two cycles, $cr(C_m \Box C_n)$, for all but finitely many n greater than each given m:

Theorem 1.1 (Glebsky and Salazar, 2004)**.** *If $n \geqslant m(m+1)$ and $m \geqslant 3$, then $cr(C_m \Box C_n) = (m-2)n$.*

In a similar line, we continue investigating the crossing number of Cartesian products of graphs, but in this case, a much larger family of graphs: namely, the Cartesian product of any vertex-transitive graph G of degree at least 3 with a cycle. We obtain lower bounds for their crossing numbers in terms of a small graph G', whose order is 2 bigger than that of G.

We only consider finite simple undirected graphs. Let G be a graph with a vertex set V and an edge set E. We only consider "good drawings" of G, in which

1. no edge crosses itself,
2. no incident edges cross,
3. no more than two edges cross at a common point,
4. edges do not cross vertices, and

5. edges that cross do so only once.

The first two types of crossings can always be eliminated, and the next three conditions are by our choice. We denote the crossing number of G for the plane by $cr(G)$. If $D(G)$ is a good drawing of G, then we denote by $v(D(G))$ the number of crossings in $D(G)$. The Cartesian product $G\square H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edge set $E(G\square H) = \{\{(x_1, y_1), (x_2, y_2)\} | x_1 = x_2$ and $y_1y_2 \in E(H)$ or $y_1 = y_2$ and $x_1x_2 \in E(G)\}$.

In 2008, Wenping et. al [7] showed the following:

Lemma 1.2 (Wenping et al., 2008)**.** *$cr(K_m\square C_n) \geqslant n \cdot cr(K_{m+2})$ for $n \geqslant 4$ and $m \geqslant 4$.*

Using their method, we prove a much more general theorem which applies to any vertex transitive graph, a graph such that for any two vertices v_1, v_2 there exists a graph automorphism on the vertex set that maps v_1 to v_2. Of course, the complete graph K_m is vertex transitive. The key observation is that, for any vertex-transitive graph G, every subgraph formed by the union of a copy of G and a copy of C_n in $G\square C_n$ are isomorphic to each other. In the following theorem, the graph G' is obtained from G by adding two vertices, fixing any vertex in G that we call v_0, and joining each of the two new vertices to v_0 and all of its neighbors as well as to each other.

Theorem 1.3 (Main Theorem)**.** *Suppose that G is a vertex-transitive graph with degree $p \geq 3$. Let us denote $|V(G)| = m$. Then for $n \geqslant 4$ and $m \geqslant 4$, we have the following lower bound for the crossing number of the Cartesian product of G and C_n:*

$$cr(G\square C_n) \geqslant \left[\frac{m}{p+1} \cdot cr(G') - (\frac{m}{p+1} - 1) \cdot cr(G)\right] \cdot n.$$

Corollary 1.4. *In the same condition,*

$$cr(G\square C_n) \geq n \cdot cr(G').$$

Observe that Wenping et al.'s result is a particular case of the above corollary. The proof is simple.

Proof. Since $m \geq p+1$ and $cr(G') \geq cr(G)$, since G is a subgraph of G', the right-hand side of this inequality is at most that of the inequality in Theorem 1. They are equal if and only if $G = K_m$, the complete graph of order $m \geqslant 4$, in which case $m = p+1$. $\square$

It is known that $cr(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ (see Richter and Sĭrán, [6]). Observe that for any vertex transitive graph G, the subgraph of G' above induced by v_0, its adjacent vertices, and the two new vertices has $K_{3,p}$ as a subgraph. Also, the intersection of this $K_{3,p}$ and the original copy of G in G' is isomorhic to the star with $p+1$ vertices. Since a star does not cross itself in a good drawing, no crossing of the $K_{3,p}$ in a good drawing occurs within the copy of G. Therefore, we have $cr(G') - cr(G) \geqslant cr(K_{3,p}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor$, and obtain the following corollary:

Corollary 1.5. *Suppose that G is a vertex-transitive graph with degree $p \geq 3$. Let us denote $|V(G)| = m$. Then for $n \geqslant 4$ and $m \geqslant 4$,*

$$cr(G\square C_n) \geqslant \frac{m}{p+1} \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \cdot n + cr(G) \cdot n.$$

This corollary shows that we can always obtain a lower bound that is better than the obvious one, $cr(G) \cdot n$. Notice that for a random graph G, we have $cr(G) \leqslant \binom{m}{2} \cdot \binom{m-2}{2} = O(m^4)$. It is of interest for future research to find a lower bound that is asymptotically greater than $cr(G) \cdot n$ by a constant multiple, or a lower bound that is $\Omega(m^4)$, if possible.

2 Proof of the Main Theorem

Proof. Parts of the proof we give here for Theorem 1.3 closely follow the counting method developed for the proof of Theorem 2.2 in Wenping et al. [8]. However, for completeness and clarification of differences in the proofs we will include the entire proof with only a few omissions and also re-introduce some borrowed notations and lemmas. In particular, if A, B are two disjoint subsets of $E(G)$, the number of crossings between edges in A and edges in B in a drawing D is denoted by $v_D(A, B)$. The number of crossings that occur between edges of A is denoted by $v_D(A)$, so that $v(D)=v_D(G)$. Also, if X is a subset of $V(G)$ or $E(G)$ for a given graph G then $G[X]$ denotes the subgraph of G induced by X. We borrow the following straightforward lemma:

Lemma 2.1 (Wenping et al., 2008)**.** *Let A, B, C be mutually disjoint subsets of $E(G)$. Then, $v_D(C, A\cup B) = v_D(C, A) + v_D(C, B)$, and $v_D(A \cup B) = v_D(A) + v_D(B) + v_D(A, B)$.*

Throughout this proof, we consider G to be a finite simple undirected graph that is vertex-transitive with a regular degree p. Let $V(G) = \{v_0, ..., v_{m-1}\}$ so that $|V(G)| = m$. Let G' be defined as in Theorem 1.3 (we may also use the apostrophe notation for a graph obtained similiarly from a vertex-transitive graph other than G). In order to simplify the notations, we define a function $f : J \times J \to \{0, 1\}$, where $J = \{0, ..., m-1\}$, related to the adjacency matrix of G as in the following:

$$f(x, y) = \begin{cases} 0, & \text{if } v_x v_y \notin E(G) \\ 1, & \text{if } v_x v_y \in E(G) \end{cases}.$$

Observe that $f(x, x) = 0$ for all $x \in J$, and if $G = K_m$ for some m, then the value of $f(x, y) = 1$ iff $x \neq y$.

Now let us consider our Cartesian product $G \Box C_n = H$. Using f, we can define the vertex set and edge set of H as in the following:

$V(H) = \{v_i^j \mid 0 \leqslant j \leqslant m-1, 0 \leqslant i \leqslant n-1\}$,

$$E(H) = \left(\bigcup_{i=0}^{n-1} \{v_j^i v_k^i \mid f(j, k) = 1\}\right) \bigcup \left(\bigcup_{i=0}^{n-1} \{v_j^{i-1} v_j^i \mid 0 \leqslant j \leqslant m-1\}\right).$$

Here and throughout Section 2, superscripts are read modulo n and subscripts are read modulo m.

We analyze H by considering it as a disjoint union of subsets of $V(H)$ and $E(H)$. For $0 \leqslant i \leqslant n-1$, let $V^i = \{v_j^i \mid 0 \leqslant j \leqslant m-1\}$, $E^i = \{v_j^i v_k^i \mid f(j, k) = 1\}$, $G^i = (V^i, E^i)$, and $M^i = \{v_j^{i-1} v_j^i \mid 0 \leqslant j \leqslant m-1\}$. Then

$E^i \cap E^j = \emptyset$ for $0 \leqslant i < j \leqslant n-1$,

$M^i \cap M^j = \emptyset$ for $0 \leqslant i < j \leqslant n-1$,

$E^i \cap M^j = \emptyset$ for $0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant n-1$,

$$E(H) = \left(\bigcup_{i=0}^{n-1} E^i\right) \cup \left(\bigcup_{i=0}^{n-1} M^i\right).$$

For any drawing D of H we have $v_D(H) = v_D\left(\left(\bigcup_{i=0}^{n-1} E^i\right) \bigcup \left(\bigcup_{i=0}^{n-1} M^i\right)\right)$. By Lemma 2.1 it follows that

$$v_D(H) = \sum_{i=0}^{n-1} v_D(E^i) + \sum_{0 \leqslant i < j \leqslant n-1} v_D(E^i, E^j) + \sum_{i=0}^{n-1} v_D(M^i) + \sum_{0 \leqslant i < j \leqslant n-1} v_D(M^i, M^j) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} v_D(E^i, M^j).$$

Furthermore, by considering the parity of n, we obtain

$$\begin{aligned} v_D(H) = &\sum_{i=0}^{n-1} v_D(E^i) + \sum_{i=0}^{n-1} v_D(M^i) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} v_D(E^i, M^j) \\ &+ \sum_{i=0}^{n-1} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} v_D(E^i, E^j) + \frac{(n+1) \mod 2}{2} \cdot \sum_{i=0}^{n-1} v_D(E^i, E^{i+\lfloor \frac{n}{2} \rfloor}) \\ &+ \sum_{i=0}^{n-1} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} v_D(M^i, M^j) + \frac{(n+1) \mod 2}{2} \cdot \sum_{i=0}^{n-1} v_D(M^i, M^{i+\lfloor \frac{n}{2} \rfloor}). \end{aligned} \tag{1}$$

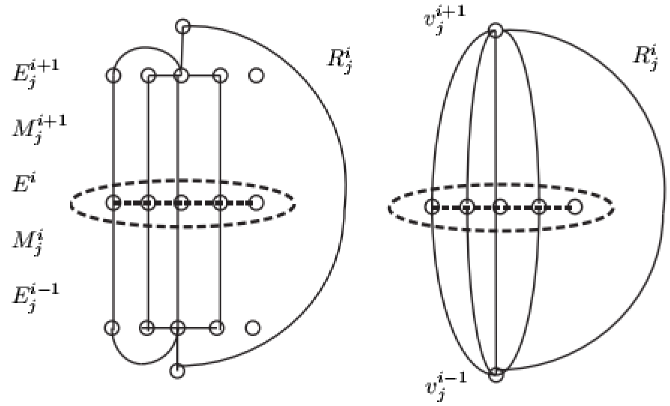

Figure 1: H_j^i and $H[E^i]'$.

Now, let us consider different subsets of $E(H)$. From here on, we fix the range of l to be $0 \leqslant m-1$, and that of k to be $0 \leqslant n-1$. Let $E_j^i = \{v_j^i v_l^i | f(j,l) = 1,\ M_j^i = \{v_l^{i-1} v_l^i | j = l$ or $f(j,l) = 1\}$, $R_j = \{v_j^0 v_j^1, v_j^1 v_j^2, ..., v_j^{n-1} v_j^0\}$, and $R_j^i = R_j \backslash \{v_j^{i-1} v_j^i, v_j^i v_j^{i+1}\} = \bigcup_{k \neq i, i+1} \{v_j^{k-1} v_j^k\}$, where $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m-1$. Then, we can conclude that $\bigcup_{j=0}^{m-1} R_j = \bigcup_{i=0}^{n-1} M^i$ and $\bigcup_{j=0}^{m-1} R_j^i = \bigcup_{k \neq i, i+1} M^k$.

For $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m-1$, let $H_j^i = H[E^i \cup E_j^{i-1} \cup M_j^i \bigcup E_j^{i+1} \cup M_j^{i+1} \cup R_j^i]$, the graph in H induced by the given union of edge sets. Then we can see from Figure 1 that H_j^i is a subdivision of $(H[E^i])'$ which is isomorphic to G'. Figure 1 shows the subgraph H_j^i and its corresponding graph $(H[E^i])'$, where the edges of E_j^{i+1}, E_j^i, and E_j^{i-1} are not drawn for clarity. Also, Figure 1 shows that in the corresponding drawing of G' for a good drawing D of H, the crossings within each of $v_D(E_j^{i-1} \cup M_j^i \cup R_j^i)$ and $v_D(E_j^{i+1} \cup M_j^{i+1} \cup R_j^i)$ need not be counted since they are either self-crossings of edges in the corresponding drawing of G' or they appear on edges emanating from the same vertex. Similarly, for $f(j,l) = 1$, the crossings between $\{v_j^{i-1} v_l^{i-1}\}$ and $\{v_j^{i+1} v_l^{i+1}\}$ need not be counted since they both emanate from v_l^i. Therefore we have:

$$
\begin{aligned}
v_D(H_j^i) \geqslant cr(G') &+ v_D\left(E_j^{i-1} \bigcup M_j^i\right) + v_D\left(E_j^{i+1} \bigcup M_j^{i+1}\right) + v_D(R_j^i) \\
&+ v_D\left(E_j^{i-1} \bigcup M_j^i, R_j^i\right) + v_D\left(E_j^{i+1} \bigcup M_j^{i+1}, R_j^i\right) \\
&+ \sum_{f(j,l)=1} v_D\left(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
v_D(H_j^i) = v_D(E^i) &+ v_D\left(E_j^{i-1} \bigcup M_j^i\right) + v_D\left(E_j^{i+1} \bigcup M_j^{i+1}\right) + v_D\left(R_j^i\right) \\
&+ v_D\left(E^i, E_j^{i-1} \bigcup M_j^i\right) + v_D\left(E^i, E_j^{i+1} \bigcup M_j^{i+1}\right) + v_D\left(E^i, R_j^i\right) \\
&+ v_D\left(E_j^{i-1} \bigcup M_j^i, E_j^{i+1} \bigcup M_j^{i+1}\right) + v_D\left(E^{i-1} \bigcup M_j^i, R_j^i\right) \\
&+ v_D\left(E_j^{i+1} \bigcup M_j^{i+1}, R_j^i\right),
\end{aligned}
$$

for $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m-1$, we obtain

$$
\begin{aligned}
cr(G') \leqslant v_D(E^i) &+ v_D\left(E^i, E_j^{i-1} \bigcup M_j^i\right) + v_D\left(E^i, E_j^{i+1} \bigcup M_j^{i+1}\right) \\
&+ v_D\left(E_j^{i-1} \bigcup M_j^i, E_j^{i+1} \bigcup M_j^{i+1}\right) + v_D\left(E^i, R_j^i\right) \\
&- \sum_{f(j,l)=1} v_D\left(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}\right).
\end{aligned}
\tag{2}
$$

So, taking the sum of both sides of (2) over all j yields

$$\begin{aligned}
m \cdot cr(G') \leqslant \sum_{j=0}^{m-1} \Bigg[& v_D(E^i) + v_D\left(E^i, E_j^{i-1} \cup M_j^i\right) + v_D\left(E^i, E_j^{i+1} \cup M_j^{i+1}\right) \\
& + v_D\left(E_j^{i-1} \cup M_j^i, E_j^{i+1} \cup M_j^{i+1}\right) + v_D\left(E^i, R_j^i\right) \\
& - \sum_{f(j,l)=1} v_D\left(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}\right) \Bigg] \\
= \sum_{j=0}^{m-1} \Bigg[& v_D(E^i) + v_D(E^i, E_j^{i-1}) + v_D(E^i, M_j^i) + v_D(E^i, E_j^{i+1}) \\
& + v_D(E^i, M_j^{i+1}) + v_D(E_j^{i-1}, E_j^{i+1}) + v_D(E_j^{i-1}, M_j^{i+1}) \\
& + v_D(M_j^i, E_j^{i+1}) + v_D(M_j^i, M_j^{i+1}) + v_D(E^i, R_j^i) \\
& - \sum_{f(j,l)=1} v_D\left(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}\right) \Bigg]. \qquad (3)
\end{aligned}$$

By observing how many times each type of crossing is counted in the above sum, we have:

$\sum_{j=0}^{m-1} v_D(E^i) = m \cdot v_D(E^i)$,

$\sum_{j=0}^{m-1} v_D(E^i, E_j^{i-1}) = 2 v_D(E^i, E^{i-1})$,

$\sum_{j=0}^{m-1} v_D(E^i, M_j^i) = (p+1) \cdot v_D(E^i, M^i)$,

$\sum_{j=0}^{m-1} v_D(E^i, E_j^{i+1}) = 2 v_D(E^i, E^{i+1})$,

$\sum_{j=0}^{m-1} v_D(E^i, M_j^{i+1}) = (p+1) \cdot v_D(E^i, M^{i+1})$,

$\sum_{j=0}^{m-1} v_D(E_j^{i-1}, M_j^{i+1}) \leqslant \sum_{j=0}^{m-1} v_D(E_j^{i-1}, M^{i+1}) = 2 v_D(E^{i-1}, M^{i+1})$,

$\sum_{j=0}^{m-1} v_D(M_j^i, E_j^{i+1}) \leqslant \sum_{j=0}^{m-1} v_D(M^i, E_j^{i+1}) = 2 v_D(M^i, E^{i+1})$,

$\sum_{j=0}^{m-1} v_D(M_j^i, M_j^{i+1}) \leqslant \sum_{j=0}^{m-1} v_D(M^i, M_j^{i+1}) = (p+1) \cdot v_D(M^i, M^{i+1})$, and

$\sum_{j=0}^{m-1} v_D(E^i, R_j^i) = v_D\left(E^i, \bigcup_{j=0}^{m-1} R_j^i\right) = v_D\left(E^i, \bigcup_{k \neq i, i+1} M^k\right)$.

These equalities and inequalities, together with the inequality (3), result in the following:

$$\begin{aligned}
m \cdot cr(G') \leqslant\ & m \cdot v_D(E^i) \\
& + 2 \cdot \left[v_D(E^i, E^{i-1}) + v_D(E^i, E^{i+1}) + v_D(E^{i-1}, M^{i+1}) + v_D(M^i, E^{i+1})\right] \\
& + (p+1) \cdot \left[v_D(E^i, M^i) + v_D(E^i, M^{i+1}) + v_D(M^i, M^{i+1})\right] \\
& + \sum_{j=0}^{m-1} \left[v_D(E_j^{i-1}, E_j^{i+1}) - \sum_{f(j,l)=1} v_D(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}) \right] \\
& + v_D(E^i, \bigcup_{k \neq i, i+1} M^k).
\end{aligned}$$

Furthermore, Wenping et al. [8] analyze different types of crossings between E^{i-1} and E^{i+1} and obtain the inequality $\sum_{j=0}^{m-1} \left[v_D\left(E_j^{i-1}, E_j^{i+1}\right) - \sum_{f(j,l)=1} v_D\left(\{v_j^{i-1} v_l^{i-1}\}, \{v_j^{i+1} v_l^{i+1}\}\right) \right] \leqslant v_D(E^{i-1}, E^{i+1})$. More pre-

cisely, they did not need to specify the condition $f(j,l)=1$, but the exactly same argument can be used to show the above inequality. The details are trivial and we omit them.
Therefore, for $0 \leqslant i \leqslant n-1$, we have

$$\begin{aligned} m \cdot cr(G') \leqslant\ & m \cdot v_D(E^i) + v_D(E^{i-1}, E^{i+1}) \\ & + 2 \cdot \left[v_D(E^i, E^{i-1}) + v_D(E^i, E^{i+1}) + v_D(E^{i-1}, M^{i+1}) + v_D(M^i, E^{i+1}) \right] \\ & + (p+1) \cdot \left[v_D(E^i, M^i) + v_D(E^i, M^{i+1}) + v_D(M^i, M^{i+1}) \right] \\ & + v_D(E^i, \bigcup_{k \neq i, i+1} M^k), \end{aligned} \tag{4}$$

so by taking the sum of both sides of (4) over all i, we get

$$\begin{aligned} mn \cdot cr(G') \leqslant\ & \sum_{i=0}^{n-1} \Bigg[m \cdot v_D(E^i) + v_D(E^{i-1}, E^{i+1}) \\ & + 2 \cdot \left[v_D(E^i, E^{i-1}) + v_D(E^i, E^{i+1}) + v_D(E^{i-1}, M^{i+1}) + v_D(M^i, E^{i+1}) \right] \\ & + (p+1) \cdot \left[v_D(E^i, M^i) + v_D(E^i, M^{i+1}) + v_D(M^i, M^{i+1}) \right] \\ & + v_D(E^i, \bigcup_{k \neq i, i+1} M^k) \Bigg] \\ & = m \cdot \sum_{i=0}^{n-1} v_D(E^i) + \sum_{i=0}^{n-1} v_D(E^i, E^{i+2}) + 4 \cdot \sum_{i=0}^{n-1} v_D(E^i, E^{i+1}) \\ & + 2 \cdot \sum_{i=0}^{n-1} v_D(E^i, M^{i+2}) + 2 \cdot \sum_{i=0}^{n-1} v_D(E^i, M^{i-1}) + (p+1) \cdot \sum_{i=0}^{n-1} v_D(M^i, M^{i+1}) \\ & + (p+1) \cdot \sum_{i=0}^{n-1} v_D(E^i, M^i) + (p+1) \cdot \sum_{i=0}^{n-1} v_D(E^i, M^{i+1}) \\ & + \sum_{i=0}^{n-1} v_D(E^i, \bigcup_{k \neq i, i+1} M^k). \end{aligned} \tag{5}$$

Comparing (5) with (1), we find that if $p \geqslant 3$ and $m, n \geqslant 4$, then

$$mn \cdot cr(G') \leqslant (m-p-1) \cdot \sum_{i=0}^{n-1} v_D(E^i) + (p+1) \cdot v_D(H).$$

Since this inequality is also true for the optimal drawing D such that $v_D(H) = cr(G \square C_n)$, we obtain

$$\begin{aligned} cr(G \square C_n) & \geqslant \frac{mn}{p+1} \cdot cr(G') - (\frac{m}{p+1} - 1) \cdot \sum_{i=0}^{n-1} v_D(E^i) \\ & \geqslant \frac{mn}{p+1} \cdot cr(G') - (\frac{m}{p+1} - 1) n \cdot cr(G). \end{aligned}$$

□

3 Conjectures and Further Research

The well-known crossing lemma states that any graph G with v vertices and $e > 4v$ edges satisfies $cr(G) \geqslant \frac{1}{64}\frac{e^3}{v^2}$. Using this, we can find that $cr(G \square C_n) \geqslant \frac{\{(0.5p+1)mn\}^3}{64(mn)^2} = O(mn)$. However, our main theorem predicts $cr(G\square C_n) \geqslant O(cr(G)\cdot mn)$, so it is a stronger result.

It would be interesting to be able to use this theorem to obtain lower bounds for $cr(G\square C_n)$ where G is the hypercube graph Q_n, the regular bi- or multi- partite graph $K_{m,m}$ or $K_{m,\ldots,m}$, the Petersen graph, or a generalized Petersen graph, all of which are vertex-transitive. Another interesting example would be the family of Cartesian products of multiple copies of isomorphic cycles, i.e. $C_n\square\ldots\square C_n$, suggested by Chiheon Kim — if possible, this would be the first general result concerning the crossing number of the Cartesian product of more than two graphs.

We can make a conjectural lower bound of $cr(G\square C_n)$ for a given vertex-transitive graph G using a conjectural value of $cr(G')$ (better yet if we can calculate $cr(G')$). For example, let Q_3 be the cubic hypercube graph, and let P be the Petersen graph. Remember that $cr(Q_3) = 0$ and $cr(P) = 2$ and that for both graphs we have $p = 3$. We propose the following conjectures based on Figure 2:

Conjecture 3.1. *$cr(Q_3') = 3$, and therefore $cr(Q_3\square C_n) \geqslant 6n$.*

Conjecture 3.2. *$cr(P') = 6$, and therefore $cr(P\square C_n) \geqslant 12n$.*

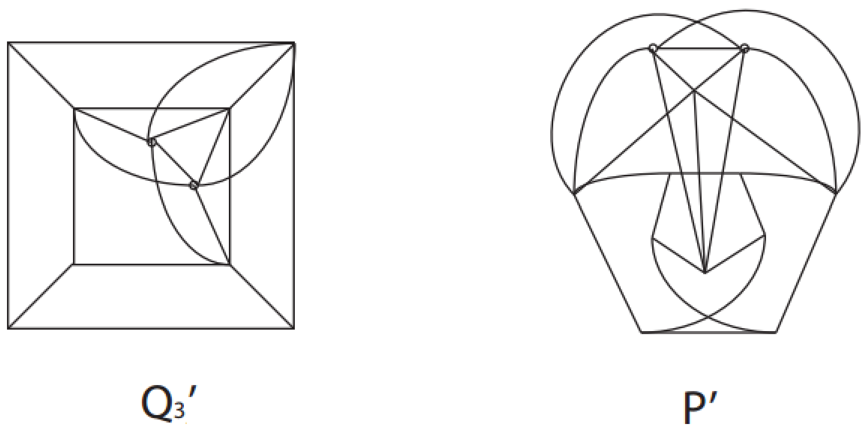

Figure 2: Q_3' and P'.

These, if true, are stronger than the lower bounds we can find by using the disjoint subcycles of G: $cr(Q_3'\square C_n) \geqslant 4n$ and $cr(P'\square C_n) \geqslant 7n$.

A different direction of research that may be fruitful is applying the theory of arrangements developed by Adamsson [1] and Adamsson and Richter [2]. For an application of the theory, see Gelebsky and Salazar [5]. It may be possible to use the theory of arrangement to calculate the crossing numbers or their lower bounds for certain small vertex-transitive graphs G and the corresponding G', such as ones mentioned above.

Finally, considering the intricate relationship between our function f and the adjacency matrix of G, it may be possible to use methods from algebraic graph theory in order to exploit subtler symmetries in non-vertex-transitive graphs and further generalize our Main Theorem.

Acknowledgements

The author is grateful to his research mentor Chiheon Kim for guiding through conducting mathematical research. He also thanks Professor Pavel Etingof and Professor Jacob Fox for suggesting an initial project concerning crossing numbers, Dr. Tanya Khovanova for general advice on mathematical research and presentation, and the MIT-PRIMES program for making this research possible.

References

[1] Adamsson J. *Ph.D. Thesis*, Carleton University, 2000.

[2] Adamsson J. and Richter R. B., *Arrangements, circular arrangements and the crossing number of* $C_7 \Box C_n$, J. Combinatorial Theory Series B **90** (2004), 21–39.

[3] Diestel R., *Graph Theory*, Springer (2012).

[4] Garey, M. R., & Johnson, D. S. *Crossing number is NP-complete.* SIAM Journal on Algebraic Discrete Methods **4** (1983), 312–316.

[5] Glebsky, L. Y., & Salazar, G. *The crossing number of* $C_m \times C_n$ *is as conjectured for* $n \geqslant m(m+1)$. J. Graph Theory **47** (2004), 53–72.

[6] Klešč M. *The crossing number of* $K_{2,3} \Box C_3$. Discrete mathematics **251** (2002), 109–117.

[7] Richter, R. B. and Sǐráň, J. *The crossing number of* $K_{3,n}$ *in a surface.* J. Graph Theory **21** (1996), 51-–54.

[8] Wenping, Z., Xiaohui, L., Yuansheng, Y., & Chengrui, D. *On the crossing numbers of* $K_m \Box C_n$ *and* $K_{m,l} \Box P_n$. Discrete Applied Mathematics **156** (2008), 1892–1907.