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- And so on …
- Each successive quotient $B_k := L_k / L_{k+1}$ keeps track of more and more non-commutativity.

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- In particular, for $k \geq 2$, each $B_k(A_n)$ has *polynomial growth*.
- The present focus is on extending these methods to characteristic p, where a geometric approach is less clear.

Lower central series of free algebras in characteristic p

Surya Bhupatiraju, William Kuszmaul, Jason Li
MIT PRIMES

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- We can take coefficients in the rational numbers $\mathbb{Q}$, integers $\mathbb{Z}$, or a finite field $\mathbb{F}_p$.

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Definition

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LOWER CENTRAL SERIES

Definition

The **lower central series filtration** of an associative algebra A

$$A = L_1 \supseteq L_2 \supseteq L_3 \supseteq \dots$$

is defined recursively by $L_1 := A, L_k := [A, L_{k-1}]$, all linear combinations of expressions $[a, b]$ for $a \in A, b \in L_{k-1}$.

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Definition

The **associated graded components** B_k to the filtration are defined as

$$B_k := L_k / L_{k+1}.$$

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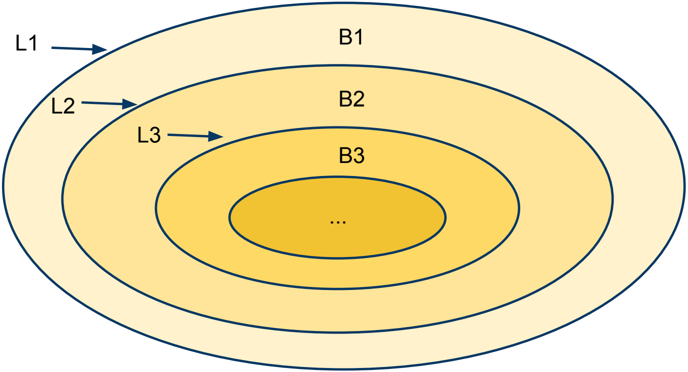

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The **multivariable Hilbert series** of V is the sum

$$h(V; t_1, \ldots, t_m) := \sum_{\mathbf{k} \in \mathbb{Z}_+^m} \dim(V_{\mathbf{k}}) t_1^{k_1} \cdots t_m^{k_m}$$

HILBERT SERIES

Example

The Hilbert series of A_2 is:

$$h(A_2; t_1, t_2) := \sum_{\mathbf{k} \in \mathbb{Z}_+^m} \binom{k+l}{k} t_1^k t_2^l = \frac{1}{1-(t_1+t_2)}$$

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  They appear to coincide.

DIFFERENCES IN HILBERT SERIES

The series $h(B_2(A_2(\mathbb{F}_p))), h(B_3(A_2(\mathbb{F}_p))), h(B_4(A_2(\mathbb{F}_p)))$ are independent of p in the range we computed.

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- Why do these changes occur?

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RESULTS FROM COMPUTATIONS

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- 2-torsion in $B_2(A_3(\mathbb{Z}))_{(2,2,2)}$,
- 3-torsion in $B_2(A_3(\mathbb{Z}))_{(3,3,3)}$,
- 2-torsion in $B_2(A_3(\mathbb{Z}))_{(4,2,2)}$, and its permutations,
- 2-torsion in $B_2(A_3(\mathbb{Z}))_{(4,4,2)}$, and its permutations,
- 2-torsion in $B_2(A_4(\mathbb{Z}))_{(2,2,2,2)}$ (3-dimensional).

Conjecture

For all a, b, c, the element:

$$v(a,b,c) = [z, z^{a-1}x^{b-1}y^{c-1}[x,y]] \in B_2(A_3(\mathbb{Z}))_{(a,b,c)},$$

is torsion, of order equal to $\gcd(a,b,c)$, and generates the torsion subgroup of $B_2(A_3(\mathbb{Z}))_{(a,b,c)}$.

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However, we still do not know v is nonzero (if $\gcd(a,b,c) > 1$)! Computation confirms this in small cases.

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