The PRIMES 2011 problem set solutions

Here are the solutions of the problems from the PRIMES 2011 problem set. Note that some solutions are just sketched (not given in full).

1. The number of all combinations out of 4 dice is $6^4 = 1296$. The number of combinations when all are different is $6 \cdot 5 \cdot 4 \cdot 3 = 360$. So the number of combinations when at least two are the same is $1296 - 360 = 936$. Out of 360, $5 \cdot 4 \cdot 3 \cdot 2 = 120$ don't contain 3. Altogether $5^4 = 625$ don't contain 3. So among the cases when at least two are equal, $625 - 120 = 505$ don't contain 3. So $936 - 505 = 431$ contain 3. So the chance of winning is $431/936$.

2. (a) Between 1 and 125, there are 25 numbers divisible by 5, 5 numbers divisible by 25, and 1 number divisible by 125. So $k = 25 + 5 + 1 = 31$.

(b) Let us compute modulo 5. We have $1 \cdot 2 \cdot 3 \cdot 4 = 4$. So the numbers between 1 and 125 not divisible by 5 contribute a factor $4^{25} = 4$. The numbers which are divisible by 5 but not by 25 contribute $4^5 = 4$. The numbers divisible by 25 contribute a 4. So altogether we get $4^3 = 4$. Thus $N = 5m + 4$.

3. These numbers are powers of 2. Indeed, sums of consecutive numbers are $S = n(n+1)/2 - m(m+1)/2 = (n-m)(n+m+1)/2$, where $n - m \geq 2$. So we get such a representation of S once we fix a factorization of $2S$ into an odd and an even number, such that the smaller factor is not 1. For S being powers of 2, such a factorization does not exist, but for all other S the factorization $2S = 2^r(2b+1)$ does the job. So there are 20 such numbers below a million (the first power of 2 bigger than a million is 2^{20}).

4. The sum

$$(\sqrt{10}+3)^{2010} + (\sqrt{10}-3)^{2010}$$

is an integer (as seen by using the binomial formula), but $\sqrt{10} - 3 < 0.2$, so the first hundred digits of the first summand are all the digits 9.

5. (b) Imagine that Manhattan is a square lattice. Introduce a coordinate system with origin at Mary's home, and the axes going diagonally. Then for each block Mary walks, the x and the y coordinate changes by 1 or -1. Returning back means that for each of them, there are exactly n copies of 1 and n copies of -1, so the answer is $\binom{2n}{n}^2$. In (a), the answer is thus $252^2 = 63,504$. For (c), note that by Stirling's formula, $\binom{2n}{n}$ is about $2^{2n}/\sqrt{\pi n}$. So the chance of return is $\binom{2n}{n}^2/4^{2n}$, which is about $1/\pi n$. For $n = 10$ it's about $1/31.4$, which is approximately 0.032.

6. The area of a triangle with integer coordinates is a half-integer, so the area S of the triangle T is rational. If the sides of the triangle are $2a, 3a, 4a$ then by Heron's formula its area is $\frac{3}{4}a^2\sqrt{15}$. So $a^2 = 4S/3\sqrt{15}$, i.e. is irrational. This is a contradiction, since by the Pythagorean theorem a^2 has to be a rational number.

7. Let p_n be the probability of the population to die out by the n-th generation (where we begin with the zeroth generation). Then $p_1 = p$, $p_2 = p^2(1-p) + p$, and in general

$$p_{n+1} = p_n^2(1-p) + p.$$

The sequence p_n by definition is increasing, so it has a limit q. Thus, $q = q^2(1-p) + p$, so $q = \frac{p}{1-p}$, or $q = 1$. If $p \geq 1/2$ then the first solution makes no sense (as we must have $q \leq 1$), so $q = 1$, and thus the probability of the population to survive indefinitely is $1 - q = 0$. If $p \leq 1/2$ then $p_n \leq \frac{p}{1-p}$ for all n (as $p_{n+1} \geq p_n$), so $q = \frac{p}{1-p}$, and the indefinite survival probability is $1 - q = \frac{1-2p}{1-p}$.

8. For twins the probability of a healthy descendant is $(1 - \frac{1}{4} \times \frac{1}{2} \times \frac{1}{10^3})^{8000}$, which is about 0.37. So the probability of disease is about 63%. For unrelated parents, the probability of a healthy descendant is $(1 - \frac{1}{4} \times 1/10^6)^{8000}$, which is about 0.998, so the probability of disease is 0.2%.

9. We have $(2m+1)^2 - 1 = 4m(m+1)$, which is divisible by 8. So any number with required property is of the form $8k+1$. Conversely, we claim that any number of the form $8k+1$ is the remainder mod 2^n of a perfect square. We prove it by induction in n starting $n = 3$. The base is clear, so we make an induction step from $n-1$ to n for $n \geq 4$. Let a be an odd number between 1 and 2^n. By the induction assumption there exists x such that $x^2 - a$ is divisible by 2^{n-1}. If it is not divisible by 2^n, then it gives remainder 2^{n-1} when divided by 2^n. Now, $(x + 2^{n-2})^2 - a = x^2 - a + 2^{n-1} + 2^{2n-4}$. The last summand is divisible by 2^n as $n \geq 4$, so $(x + 2^{n-2})^2 - a$ is divisible by 2^n, as desired.

10. (a) We have $a_n = 2^n$ for $n = 0, ..., k-1$. If $n \geq k$ then a sequence satisfying our condition ends with a 1, or with 10, or with 100, or with 100..0 ($k-1$ zeros). So

$$a_n = a_{n-1} + ... + a_{n-k}.$$

(b) By (a) the first values of $a_n(3)$ are:
1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504.
So $a_{10}(3) = 504$.
(c) The characteristic equation of the recursion is

$$t^k = t^{k-1} + ... + 1.$$

This equation has a unique positive solution $1 < T < 2$, which is a simple root, as seen by setting $t = 1/x$ (this gives $1 = x + x^2 + ... + x^n$). Moreover, for any complex root z of the equation, one has $|z| < T$ (as again is seen by setting $t = 1/x$). This implies that the general solution of the recursion is of the form

$$y_n = CT^n + \varepsilon_n,$$

where ε_n/T^n goes to 0. It remains to show that in our case $C > 0$. To see this, it suffices to note that for any solution y_n of the recursion, there is a constant M such that $|y_n| \leq Ma_n$.

Thus, $a_n \cong CT^n$ and $p_n \cong C(T/2)^n$. So $b = T/2$ and thus $b < 1$

(d) For $k = 2$, $a_n(2)$ is the Fibonacci sequence, and $T^2 = T + 1$, so T is the golden ratio, and $b_2 = \frac{\sqrt{5}+1}{4}$.

11. (a) Assume the contrary, and let V be a 2-dimensional space with the required property. Let $A, B \in V$ be non-proportional. Then $\det(At + B)$ is a polynomial of degree n. This degree is odd, so f has a real root. For this root t, $At + B$ is not invertible, a contradiction.

(b) Pick $V \subset M(r)$ of dimension $f(r)$ and take $V' \subset M(rs)$ consisting of $A \oplus ... \oplus A$ (s times), $A \in V$.

(c) If $V \subset M(n)$ is an $n+1$-dimensional space, then it has a nonzero element with zero first row.

(d) $f(2) = 2$ (V=complex numbers).

(e) $f(4) = 4$ (V=quaternions).

(f) $f(8) = 8$ (V=octonions).

(g) For $\mathbb{C}$, $f(n) = 1$ for all n, since $\det(At + B)$ always has a root (similar to (a)). For $\mathbb{Q}$, $f(n) = n$; indeed, $f(n) \leq n$ by (c), and for n-dimensional V we can take $\mathbb{Q}(2^{1/n})$ acting on itself by left multiplication (in some basis, e.g. $2^{j/n}$, $j = 0, ..., n-1$).

12. (a) The sequence is decreasing, so has a limit x such that $x = x - x^{m+1}$, so $x = 0$.

(b,c) Compare the sequence x_n to $y_n = Cn^{-b}$, $b > 0, C > 0$. The sequence y_n satisfies the recursion

$$y_{n+1} = C((y_n/C)^{-1/b} + 1)^{-b},$$

or

$$y_{n+1} = \frac{y_n}{(1 + (y_n/C)^{1/b})^b} = y_n - bC^{-1/b}y_n^{1+1/b} + ...$$

So one should expect that $1 + 1/a = m + 1$, or $a = \frac{1}{m}$. Also, one should expect that $K^{-m}/m = 1$, so $K = m^{-1/m}$. To prove that this is indeed the case for a, let us write y_n as $y_n(b, C)$ and squeeze the sequence x_n between two (possibly shifted) sequences of the form $y_n(a_1, C_1)$ and $y_n(a_2, C_2)$ for any $a_1 < \frac{1}{m} < a_2$. Once this is done, we similarly squeeze

x_n between two (possibly shifted) sequences $y_n(a, C_1)$ and $y_n(a, C_2)$ for any $C_1 < m^{-1/m} < C_2$, proving (c).