The Pak–Postnikov and Naruse skew hook length formulas: a new proof

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/yd2023.pdf

Young diagrams

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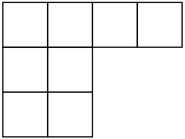
- **Example:** If $\lambda = (4, 2, 2, 0, 0, 0, \ldots) = (4, 2, 2)$ (we omit zeroes), then

$Y(\lambda) =$  .

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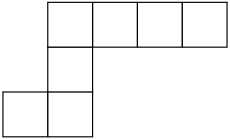
- **Example:** If $\lambda = (5,2,2)$ and $\mu = (1,1)$, then

$Y(\lambda/\mu) =$  .

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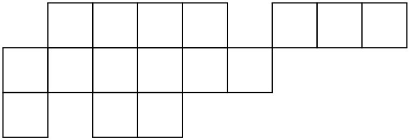
- More generally, any set of (square) cells is called a *diagram*.
- **Example:**



- Given a diagram D, we can fill it with the numbers $1, 2, \ldots, n$. Such a filling is called a *standard tableau* (of shape D) if
  - each of the numbers $1, 2, \ldots, n$ appears exactly once;
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Standard tableaux

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- Likewise $\mathrm{SYT}(\lambda/\mu)$.
- **Example:** If $\lambda = (5, 4, 3, 3)$ and $\mu = (2, 1, 1)$, then

		1	3	9
	2	4	10	
	5	6		
7	8	11		

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- **Question:** Given a diagram D, how many standard tableaux of shape D exist?

Hooks

- For $D = Y(\lambda)$, the classical *hook length formula* of Frame, Robinson and Thrall (1953) gives a beautiful answer in terms of the **hooks** of λ.

Hooks

- If $c = (i, j)$ is a cell of a Young diagram $Y(\lambda)$, we let the *hook $H_\lambda(c)$* be

$$\begin{aligned} &\{\text{all cells of } Y(\lambda) \text{ lying due east of } c\} \\ &\qquad \cup \{\text{all cells of } Y(\lambda) \text{ lying due south of } c\} \cup \{c\} \\ &= \{(i,k) \in Y(\lambda) \mid k \geq j\} \cup \{(k,j) \in Y(\lambda) \mid k \geq i\}. \end{aligned}$$

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- **Example:** If $\lambda = (4, 3, 3, 2)$, then

$H_\lambda(2,2) =$

	*	*	
	*		
	*		

and $h_\lambda(2,2) = 4$.

- The original *hook length formula* says that

$$|\mathrm{SYT}(\lambda)| = \frac{n!}{\prod_{c \in Y(\lambda)} h_\lambda(c)},$$

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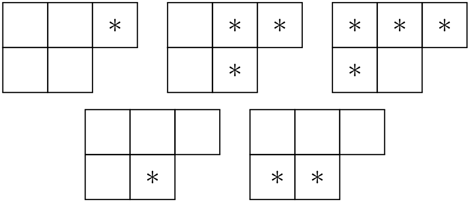
- **Example:** If $\lambda = (3, 2)$, then

$$|\,\mathrm{SYT}(\lambda)\,| = \frac{5!}{1 \cdot 3 \cdot 4 \cdot 1 \cdot 2} = 5.$$

Here are the hooks of all five cells:



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  Here is $\mathrm{SYT}(\lambda)$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\,,$$

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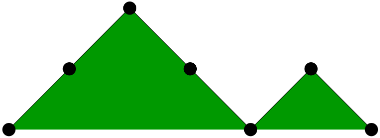
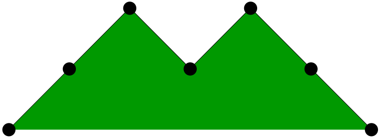
- **Example:** The number of Dyck paths from $(0,0)$ to $(2n,0)$ is the n-th *Catalan number* $C_n = \dfrac{(2n)!}{n!\,(n+1)!}$.
  This follows from the hook length formula, applied to $\lambda = (n,n)$, and a simple bijection $\{\text{Dyck paths}\} \to \mathrm{SYT}\,(\lambda)$:



$\longleftrightarrow$

1	2	5
3	4	6

.



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- *Naruse's skew hook length formula* (Naruse, 2014) expresses $|\mathrm{SYT}(\lambda/\mu)|$ in terms of excitations.

Excited moves

- An *excited move* for a cell $c = (i, j) \in D$ means moving this cell from (i, j) to $(i+1, j+1)$.
  This is allowed only if the three cells marked $\times$ (that is, $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$) are not in D.

$$\begin{array}{cc} \boxed{c} & \times \\ \times & \times \end{array} \quad \to \quad \begin{array}{cc} \times & \times \\ \times & \boxed{c} \end{array}$$

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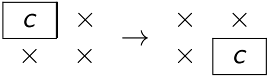

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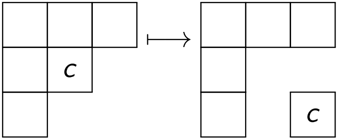

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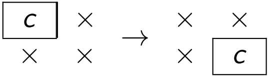

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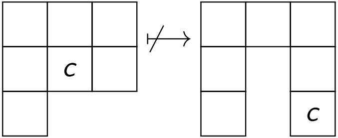

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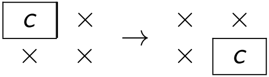

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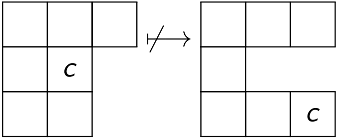

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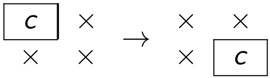

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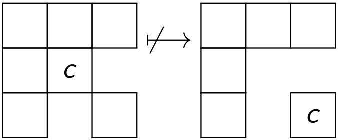

Excitations

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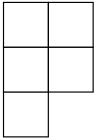
- **Example:** Original diagram D:



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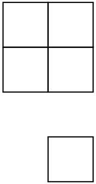
Example: After a single excited move:



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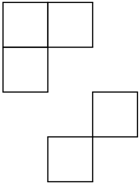
Example: After two excited moves:



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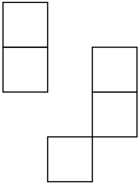
Example: After three excited moves:



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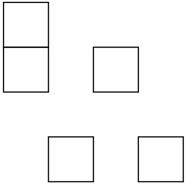
Example: After four excited moves:



- An *excitation* of a diagram D is a diagram obtained from D by a sequence of excited moves.

- Now, for two partitions λ and μ, we define $\mathcal{E}(\lambda/\mu)$ to be the set of all excitations E of $Y(\mu)$ that satisfy $E \subseteq Y(\lambda)$.

- *Naruse's skew hook length formula* says that

$$|\mathrm{SYT}(\lambda/\mu)| = n! \sum_{E \in \mathcal{E}(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c)}$$

if λ and μ are two partitions with $\mu \subseteq \lambda$ with $|Y(\lambda/\mu)| = n$.

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- **Example:** If $\lambda = (2,2,2)$ and $\mu = (1,1)$, then

$$\mathcal{E}(\lambda/\mu) = \left\{ \begin{array}{|c|c|}\hline * & \\ \hline * & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|}\hline * & \\ \hline & \\ \hline & * \\ \hline \end{array}, \begin{array}{|c|c|}\hline & \\ \hline & * \\ \hline & * \\ \hline \end{array} \right\}.$$

  Thus,

$$|\mathrm{SYT}(\lambda/\mu)| = 4! \cdot \left(\frac{1}{3 \cdot 2 \cdot 1 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 3 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 3 \cdot 4} \right) = 3$$

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- Known proofs use algebraic geometry (Naruse) or complicated combinatorics (Morales/Pak/Panova and Konvalinka).

The Pak–Postnikov generalization

- In 2001, Pak and Postnikov generalized the classical hook length formula in a different direction.

The Pak–Postnikov generalization

- If T is a standard tableau (of any shape), and if k is a positive integer, then $c_T(k)$ shall denote the difference $j - i$, where (i, j) is the cell of T that contains the entry k.
- **Example:** If $T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$, then

$$\begin{aligned} c_T(1) &= 1 - 1 = 0, \\ c_T(2) &= 1 - 2 = -1, \\ c_T(3) &= 2 - 1 = 1, \\ c_T(4) &= 3 - 1 = 2, \\ c_T(5) &= 2 - 2 = 0. \end{aligned}$$

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- Let $\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots$ be commuting indeterminates.

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- Let $\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots$ be commuting indeterminates.
- For any cell $c = (i, j)$ of $Y(\lambda)$, we define the *algebraic hook length* $h_\lambda(c; z)$ by

$$h_\lambda(c; z) := \sum_{(i,j) \in H_\lambda(c)} z_{j-i}.$$

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- The *Pak-Postnikov generalization of the hook length formula* states that

$$\sum_{T \in \mathrm{SYT}(\lambda)} z_T = \prod_{c \in Y(\lambda)} \frac{1}{h_\lambda(c; z)}.$$

- **Example:** For $\lambda = (2, 1)$, we have

$$\mathrm{SYT}(\lambda) = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right\},$$ so the formula becomes

$$\frac{1}{z_{-1}(z_{-1}+z_1)(z_{-1}+z_1+z_0)} + \frac{1}{z_1(z_1+z_{-1})(z_1+z_{-1}+z_0)}$$
$$= \frac{1}{(z_1+z_{-1}+z_0)\, z_1 z_{-1}}.$$

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- Known proofs involve polytopes (Pak/Postnikov) or P-partitions and tropical RSK (Hopkins).

Merging the streams: skew Pak–Postnikov

- We propose a generalization of the Pak–Postnikov formula to skew diagrams, thus extending Naruse's hook length formula as well.

- **Main theorem.** Let λ and μ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has n cells.
  Define z_T for $T \in \mathrm{SYT}(\lambda/\mu)$ as before.
  Define $h_\lambda(c; z)$ for $c \in Y(\lambda)$ as before (this does not depend on μ!).

- **Main theorem.** Let λ and μ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has n cells. Then,

$$\sum_{T \in \mathrm{SYT}(\lambda/\mu)} z_T = \sum_{E \in \mathcal{E}(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c; z)}.$$

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- **Example:** For $\lambda = (2,2)$ and $\mu = (1)$, we have

$$\mathrm{SYT}(\lambda/\mu) = \left\{ \begin{array}{cc} & 1 \\ 2 & 3 \end{array}, \quad \begin{array}{cc} & 2 \\ 1 & 3 \end{array} \right\} \quad \text{and} \quad \mathcal{E}(\lambda/\mu) = \left\{ \begin{array}{cc} * & \\ & \end{array}, \quad \begin{array}{cc} & \\ & * \end{array} \right\},$$

so the formula becomes

$$\frac{1}{z_0 \cdot (z_0 + z_{-1}) \cdot (z_0 + z_{-1} + z_1)} + \frac{1}{z_0 \cdot (z_0 + z_1) \cdot (z_0 + z_1 + z_{-1})} =$$
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- This was first observed by Grinberg. An intricate combinatorial proof was sketched by Konvalinka in 2019.

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- Induct on $|Y(\lambda/\mu)|$, increasing μ by one cell in the induction step.

Proof idea: the Konvalinka recursion, 1

- Let $f(\lambda/\mu) = \sum_{T \in \mathrm{SYT}(\lambda)} z_T$.

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where T' is the same tableau as T, with the entry 1 removed and all other entries decreased by 1.

- **Example:** Let $\lambda = (3,3,2)$ and $\mu = (2,1)$.

If T is ...

$$\begin{array}{ccc} & & \boxed{2} \\ & \boxed{1} & \boxed{3} \\ \boxed{4} & \boxed{5} & \end{array} \in \mathrm{SYT}(\lambda/\mu)$$

then T' is ...

$$\begin{array}{ccc} & & \boxed{1} \\ & & \boxed{2} \\ \boxed{3} & \boxed{4} & \end{array} \in \mathrm{SYT}(\lambda/\nu)$$

for $\nu = (2,2)$. Thus, $z_T = \frac{1}{z_{-1}+z_{-2}+z_1+z_2+z_0} \cdot z_{T'}$.

Proof idea: the Konvalinka recursion, 2

- Thus we get a recurrence for $f(\lambda/\mu)$:

$$f(\lambda/\mu) = \frac{1}{\sum\limits_{(i,j)\in Y(\lambda/\mu)} z_{j-i}} \cdot \sum_{\mu \lessdot \nu \subseteq \lambda} f(\lambda/\nu).$$

- Here, $\mu \lessdot \nu$ means that the partition ν is obtained by adding 1 to some entry of μ.

- Thus we get a recurrence for $f(\lambda/\mu)$:

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- Here, $\mu \lessdot \nu$ means that the partition ν is obtained by adding 1 to some entry of μ.
- The induction step thus reduces to the following claim:
- **Konvalinka recursion.** Let λ/μ be any skew partition, and let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be two infinite families of commuting indeterminates. Then,

$$\left(\sum_{\nexists i:\ \lambda_k - k = \mu_i - i} x_k + \sum_{\nexists j:\ \lambda^t_p - p = \mu^t_j - j} y_p \right) \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (x_i + y_j)$$
$$= \sum_{\mu \lessdot \nu \subseteq \lambda} \sum_{D\in\mathcal{E}(\lambda/\nu)} \prod_{(i,j)\in D} (x_i + y_j).$$

Proof ingredient 1: Flagged SSYTs, 1

- Let D be a diagram. A *semistandard tableau* (of shape D) means a filling of the cells of D with positive integers such that
  - the numbers **weakly** increase along each row,
  - the numbers **strictly** increase down each column.
- **Example:** Here is a semistandard tableau for $\mu = (4, 3, 1)$:

1	1	1	2
2	3	3	
4			

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- A *flagging* means a sequence $\mathbf{b} := (b_1, b_2, b_3, \ldots)$ of positive integers.

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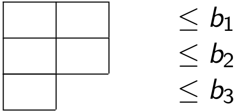

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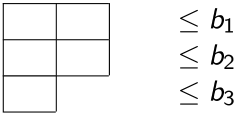


- For two partitions λ and μ, we define $\mathcal{F}(\lambda/\mu)$ to be the set of flagged semistandard tableaux of shape $(\mu, \mathbf{b})$, where $\mathbf{b} := (b_1, b_2, b_3, \ldots)$ with

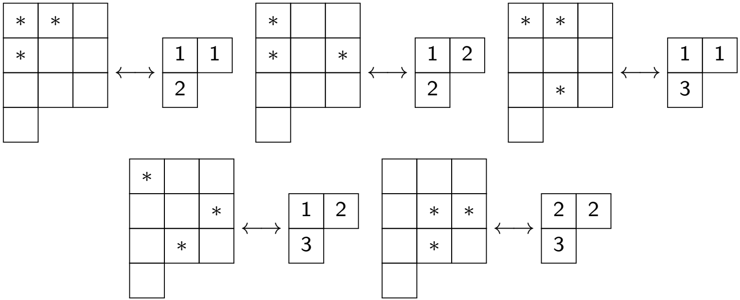
$$b_i := \max\{k \geq i \mid \lambda_k - k \geq \mu_i - i\} \qquad \text{for all } i \geq 1.$$

- Now, there is a bijection from $\mathcal{E}(\lambda/\mu)$ to $\mathcal{F}(\lambda/\mu)$, defined as follows: Each excitation $D \in \mathcal{E}(\lambda/\mu)$ is sent to the flagged semistandard tableau T of shape $(\mu, \mathbf{b})$, where

  $$T(i,j) = i + (\# \text{ of excited moves that cell } (i,j) \text{ makes in } D).$$

  Here $T(i,j)$ means the entry of T in cell (i,j).

- **Example:** For $\lambda = (3,3,3,1)$ and $\mu = (2,1)$:



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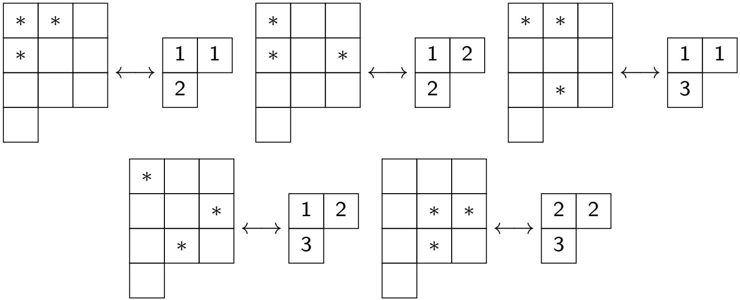

- Thus, we can work with flagged SSYTs instead of excited diagrams.

- **Theorem (generalized Jacobi–Trudi formula).** Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ be two partitions. Let $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$ be positive integers. Let $u_{i,j}$ be a variable for each pair $(i,j) \in \mathbb{Z}^2$.
  Then,

$$\sum_{\substack{T \text{ is a semistandard tableau} \\ \text{of shape } Y(\lambda/\mu); \\ a_i \leq T(i,j) \leq b_i \text{ for all } (i,j)}} \prod_{(i,j) \in Y(\lambda/\mu)} u_{j-i,\ T(i,j)}$$

$$= \det \left(\sum_{a_i \leq t_{\mu_i+1} \leq t_{\mu_i+2} \leq \cdots \leq t_{\lambda_j} \leq b_j} \prod_{c=\mu_i+1}^{\lambda_j} u_{c-i,\ t_c} \right)_{i,j \in [k]} .$$

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- This is implicit in a preprint of Gessel and Viennot 1989.

- If $\mu = (0, 0, \ldots, 0)$ and all a_i are 0 as well, and if $u_{i,j} = x_j + y_{i+j}$, and if we rename λ as μ, then the left hand side here becomes

$$\sum_{T \in \mathrm{FSSYT}(\mu, \mathbf{b})} \prod_{(i,j) \in Y(\mu)} \left(x_{T(i,j)} + y_{T(i,j)+j-i} \right),$$

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which equals the

$$\sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (x_i + y_j)$$

in the Konvalinka recursion.

Proof ingredient 3: a determinantal identity

- Jacobi–Trudi transforms both sides of the Konvalinka recursion into sums of determinants.
- After some nontrivial work, it becomes an easy determinantal identity:

Proof ingredient 3: a determinantal identity

- **Theorem.** Let M and N be two $n \times n$-matrices. Then,

$$\sum_{k=1}^{n} \det(M \text{ with its } k\text{-th row replaced by the } k\text{-th row of } N)$$
$$= \sum_{k=1}^{n} \det(M \text{ with its } k\text{-th column replaced by the } k\text{-th column of } N).$$

- **Example:**

$$\det \begin{pmatrix} A & B & C \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & c \\ A' & B' & C' \\ a'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ A'' & B'' & C'' \end{pmatrix}$$
$$= \det \begin{pmatrix} A & b & c \\ A' & b' & c' \\ A'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & B & c \\ a' & B' & c' \\ a'' & B'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & C \\ a' & b' & C' \\ a'' & b'' & C'' \end{pmatrix}.$$

Proof ingredient 4: Combinatorial lemmas

- Two of the lemmas used along the way:
- **Lemma 1.** Let λ be a partition. Let λ^t be its conjugate (i.e., Young diagram flipped across the main diagonal).
  Then, the sets

$$\{\lambda_i - i \mid i \in \mathbb{N}\} \quad \text{and} \quad \left\{j - \lambda_j^t - 1 \mid j \in \mathbb{N}\right\}$$

  are disjoint and their union is $\mathbb{Z}$.
- **Lemma 2.** Let $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ be the flagging of λ/μ.
  Let μ^{+i} be the partition obtained from μ by increasing the i-th entry by 1.
  Let $\mathbf{b}^{*i} = \left(b_1^{*i}, b_2^{*i}, b_3^{*i}, \ldots\right)$ be the flagging induced by λ/μ^{+i}.
  Then:

$$-1 \leq b_i^{*i} - b_i \leq 0, \qquad \text{and} \qquad b_k^{*i} = b_k \text{ for all } k \neq i.$$

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