Cayley's formula

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Primes-Switzerland

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Labeled Trees

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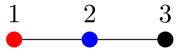

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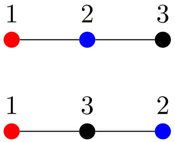

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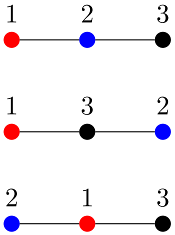

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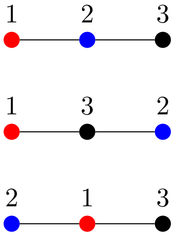

We thus have $T_3 = 3$.

Cayley's formula

Theorem (Cayley)

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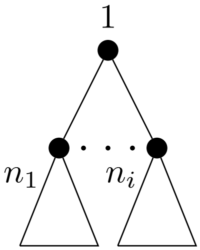
This talk: Another proof of this theorem!

Direct Proof by Induction?



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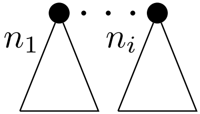

Labeled Forest

Generalization of T_n:

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Let $T_{n,k}$ be the number of forest where

- n is the number of labeled vertices,
- and vertices $1, \ldots, k$ all belong to different trees.

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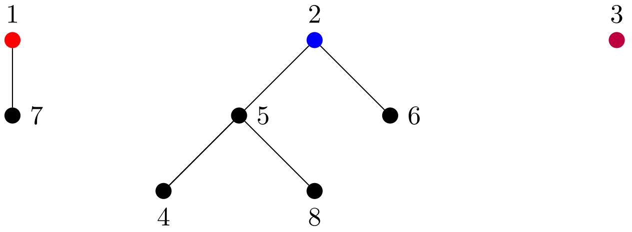

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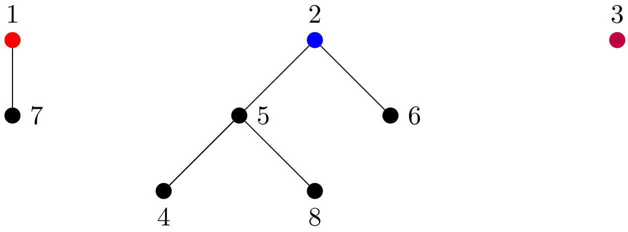
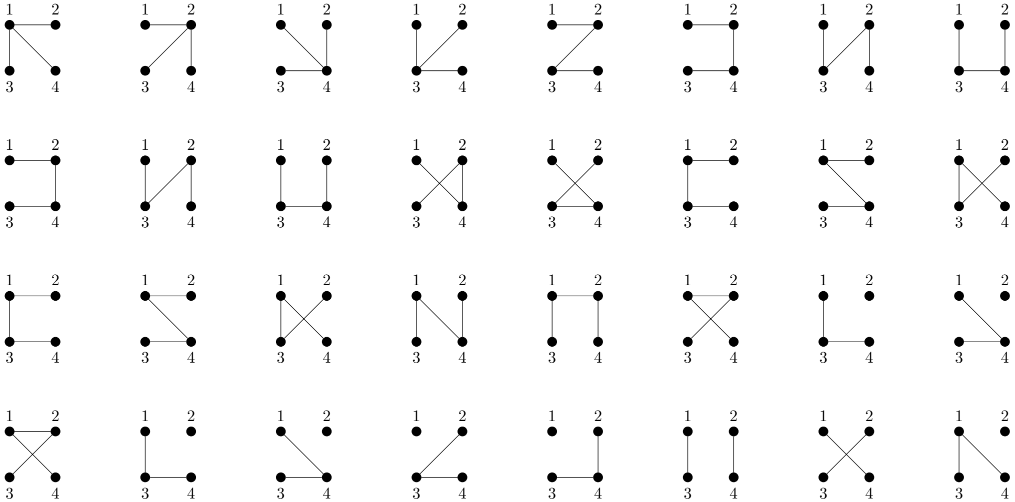


In particular, $T_n = T_{n,1}$.

Example: $T_{4,k}$



Proof Structure

Step A: Show for $n \in \mathbb{N}$ and $k \leq n$:

$$T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,(k-1)+i}$$

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Step A: Show for $n \in \mathbb{N}$ and $k \leq n$:

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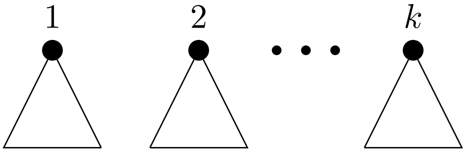
Step C: Deduce Caley's formula.

Step A

Lemma $T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,(k-1)+i}$ for all $n \in \mathbb{N}$ and $k \leq n$



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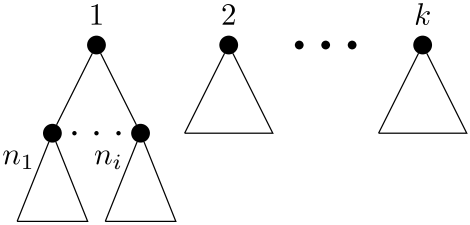
(1) **Case distinction** on the number $i \in \{0, n-k\}$ of neighbors of the vertex 1.



$$T_{n,k} = \overbrace{\sum_{i=0}^{n-k}}^{(1)}$$

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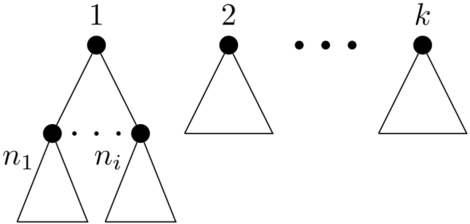
(2) There are $\binom{n-k}{i}$ options to label the i neighbors.



$$T_{n,k} = \overbrace{\sum_{i=0}^{n-k}}^{(1)} \overbrace{\binom{n-k}{i}}^{(2)}$$

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(1) **Case distinction** on the number $i \in \{0, n-k\}$ of neighbors of the vertex 1.

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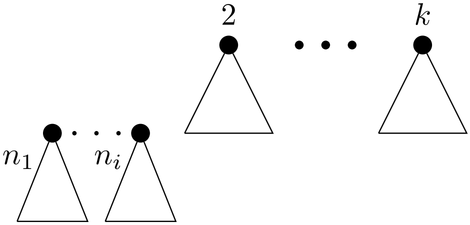
(3) Without 1 the rest of the forest consists of $i + k - 1$ labeled trees.



$$T_{n,k} = \overbrace{\sum_{i=0}^{n-k}}^{(1)} \overbrace{\binom{n-k}{i}}^{(2)} \overbrace{T_{n-1,(k-1)+i}}^{(3)}$$

Step B

Lemma $T_{n,k} = kn^{n-k-1}$ for all $n \in \mathbb{N}$ and $k \leq n$

Proof by Induction on n:
Base $n = 1$**:**

$$T_{1,1} = 1 = 1 \cdot 1^{1-1-1}$$

Step B: Induction Step

Step $n-1 \longrightarrow n$:

$T_{n,k}$

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$$T_{n,k} \overset{\textbf{Step A}}{=} \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,(k-1)+i}$$

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Step $n-1 \longrightarrow n$**:**

(*) $\sum_{j=0}^{m} f(j) = \sum_{j=0}^{m} f(m-j)$

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$$\overset{(*)}{=} \sum_{i=0}^{n-k} \binom{n-k}{(n-k)-i} T_{n-1,(k-1)+(n-k)-i}$$

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$$\overset{\textbf{IH}}{=} \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1-i)(n-1)^{i-1}$$

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$$\overset{\text{IH}}{=} \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1-i)(n-1)^{i-1}$$

$$= \underbrace{\sum_{i=0}^{n-k} \binom{n-k}{i} (n-1)(n-1)^{i-1}}_{(1)} - \underbrace{\sum_{i=0}^{n-k} \binom{n-k}{i} i(n-1)^{i-1}}_{(2)}$$

Step B: Simplify (1)

$$(1) \quad = \quad \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1)(n-1)^{i-1}$$

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(1) \quad &= \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1)(n-1)^{i-1} \\
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Step B: Simplify (2)

$$(2) \quad = \quad \sum_{i=0}^{n-k} \binom{n-k}{i} i(n-1)^{i-1}$$

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&\overset{Binom.}{=} (n-k)(n-1+1)^{n-k-1} = (n-k)n^{n-k-1}
\end{aligned}$$

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Step $n-1 \longrightarrow n$:

$$T_{n,k} = \underbrace{\sum_{i=0}^{n-k} \binom{n-k}{i} (n-1)(n-1)^{i-1}}_{(1)} - \underbrace{\sum_{i=0}^{n-k} \binom{n-k}{i} i(n-1)^{i-1}}_{(2)}$$

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$\square$

Step C: Caley's formula

Lemma $T_n = n^{n-2}$ for all $n \in \mathbb{N}$

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Proof:

$$T_n = T_{n,1} \overset{\text{Step B}}{=} 1n^{n-1-1} = n^{n-2}$$

$\square$