Kruskal's Greedy Algorithm

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Problem

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Find a spanning-tree T such that the sum of the weight of all edges in T is minimal.

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- What is a Graph?
- What is a Tree?

What is a graph?

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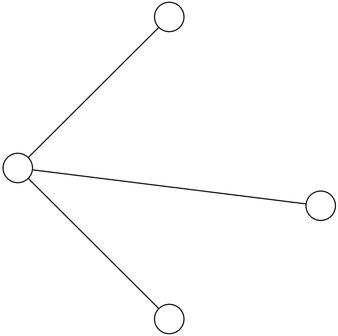
A set of vertices connected by edges



The edges don't have to be straight, and they're allowed to intersect

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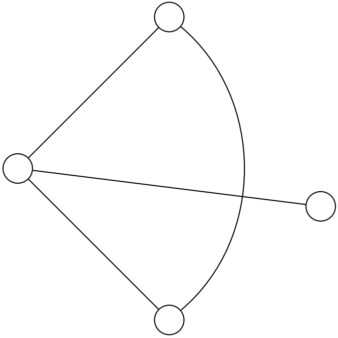

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What is a tree?

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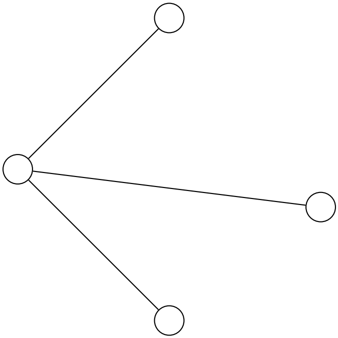
A connected graph without any cycles



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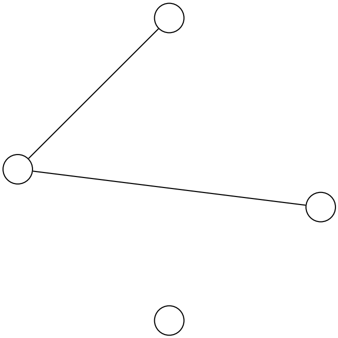

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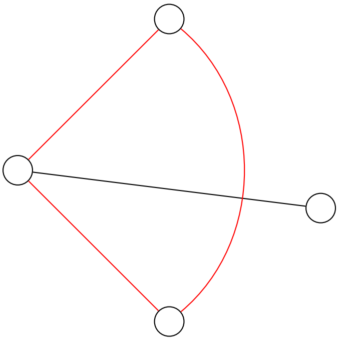
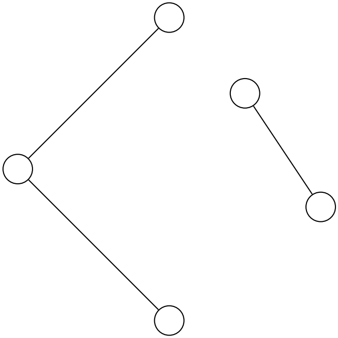


What is a forest?

What is a forest?

Like a tree, but it doesn't need to be connected.
$\Rightarrow$ A graph without a cycle



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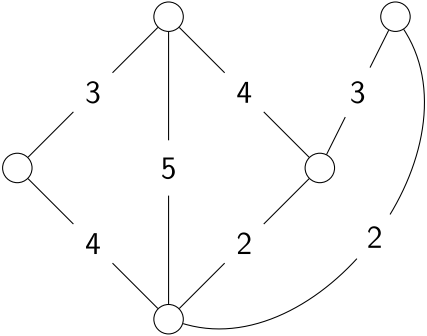

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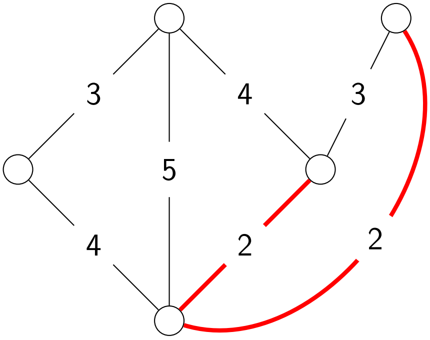

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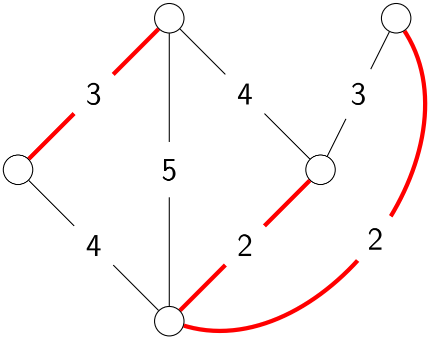

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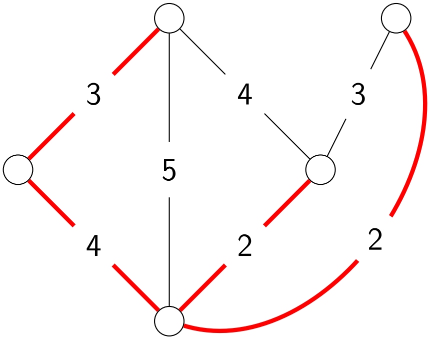

That is Kruskal's algorithm

Always add a edge with the lowest weight, which isn't already in the forest and doesn't create a cycle.

Lemma 1

A graph is a tree on n vertices $\Rightarrow$ it has $n-1$ edges

Induction on the number of vertices.

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Trivial case: 1 vertex

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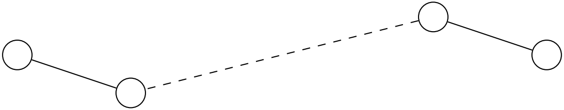
Suppose it holds for n vertices. Let T be a tree on $n+1$ vertices.
Find a longest path.



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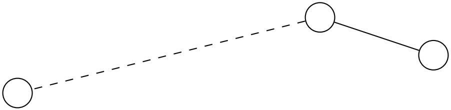
Find a longest path. Remove one of the vertices where it ends, you get a new tree T'

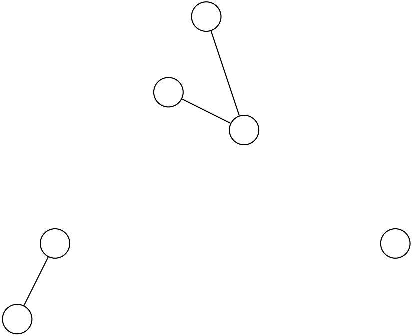


T' is a tree on n vertices $\Rightarrow$ (inductive hypothesis) it has $n-1$ edges. $\Rightarrow T$ has n edges.
$\square$

Lemma 2

A graph is a forest on n vertices with k components $\Rightarrow$ it has $n-k$ edges.

Apply Lemma 1 on all components



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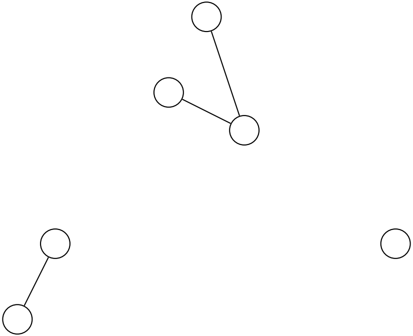

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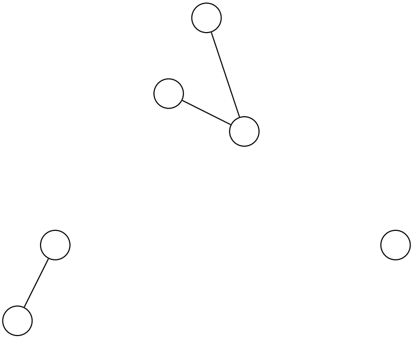

by Lemma 1 : in every component $\#$ edges $=$ $\#$ vertices - 1

$\Rightarrow$ $\#$ all edges $=$ $\#$ all vertices - k

$\square$

Lemma 3

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$\Rightarrow$ There exists $e \in F'$ such that $F \cup e$ is still a forest.

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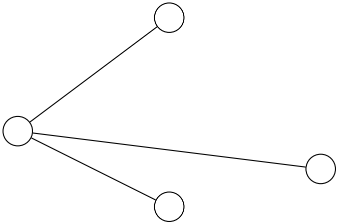

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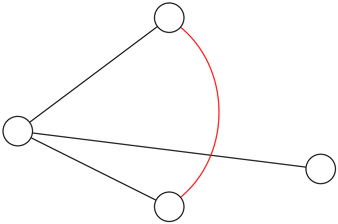

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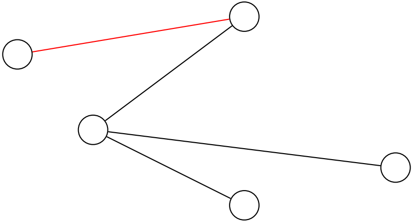

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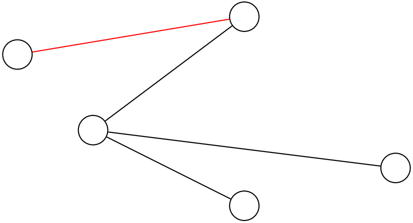

$\Rightarrow$ $\#$ components in $F \leq$ $\#$components in F'.

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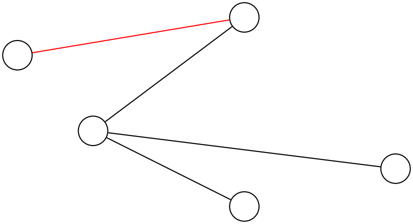

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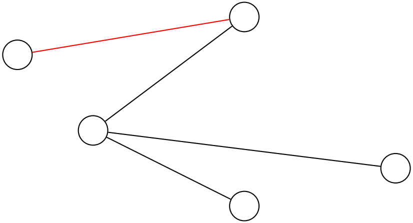

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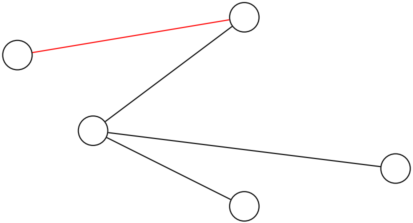

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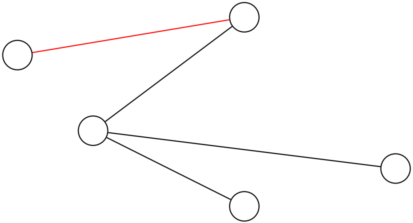

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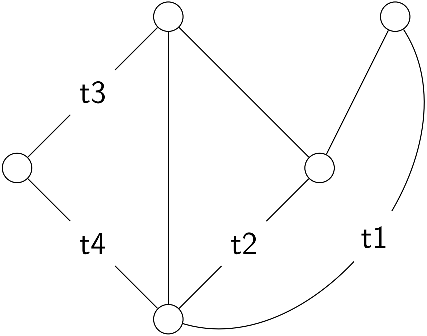
contradiction! $\square$

Graph G, trees T and H, with $\omega(T) > \omega(H)$
Sort the edges in H and T by weight. $\Rightarrow h_1, ..., h_n$ and $t_1, ..., t_n$
H_i and T_i are the forests made by the first i edges of H and T



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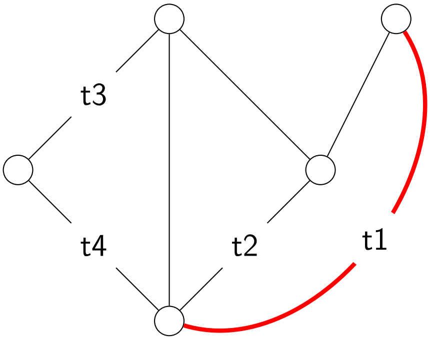

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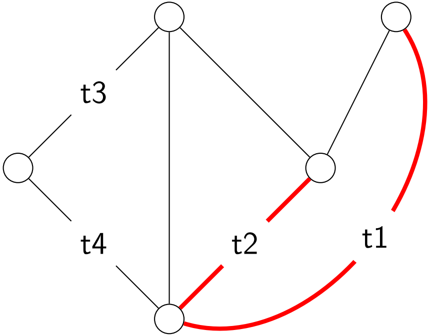

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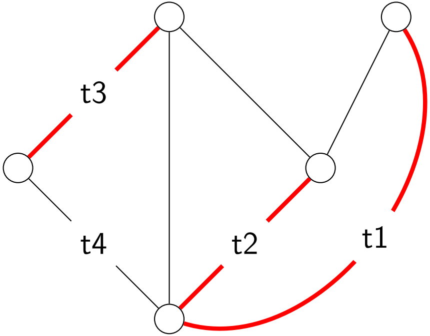

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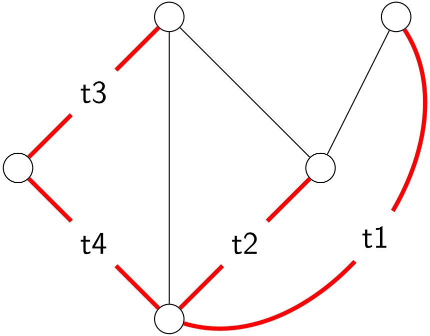

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$\Rightarrow$ contradiction $\Rightarrow$ Kruskal's Algorithm finds a minimum weight spanning-tree!