# Ramification of weak Arthur packets for $p$-adic groups (joint work w. Emile Okada)

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1. Quick and biased review of local Arthur packets (can skip)
2. Second introduction: A cheap ideology
3. Weak Arthur packets
4. Weak sphericity (main theorem)
5. Nilpotent cone geometry
   - Relative special pieces
   - Lusztig's canonical quotients

# Some global matters

- For exposition, $\mathbf{G}$ is a semisimple group defined over a number field $k$. For all but finitely many ($p$-adic) completions $k < k_v$, the locally compact group $\mathbf{G}(k_v)$ has a well-defined (hyperspecial) maximal compact subgroup $K_v = \mathbf{G}(\mathfrak{O}_v)$, where $\mathfrak{O}_v < k_v$ is the $p$-adic ring of integers.
- An irreducible representation smooth representation of the adélic group $\mathbf{G}(\mathbb{A}_k) = \prod'_v \mathbf{G}(k_v)$ is an infinite tensor products of the form

$$\pi = \otimes_v \pi_v \ ,$$

  where each $\pi_v$ is a smooth irreducible $\mathbf{G}(k_v)$-representation, so that all but finitely many of them are *spherical*.
- A spherical (or, unramified) $\pi_v$ is one that has a non-zero $K_v$-invariant vector.

# Some global matters

- Two such representations $\pi = \otimes_v \pi_v, \pi' = \otimes_v \pi'_v$ of are said to be *near-equivalent*, when for all but finitely many $v$, the (spherical) representations $\pi_v \cong \pi'_v$ are isomorphic.
- When classifying *automorphic* representations of $\mathbf{G}(\mathbb{A}_k)$, the near-equivalence relation seems to be natural.
- Indeed, (at least) for $\mathbf{G} = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$, the celebrated endoscopic project of Arthur and others gave a description of all near-equivalence classes in a suitable automorphic space. (sorry for lack of details...)
- Essentially, these are the *(global) Arthur packets* $\{\Pi_\Psi\}_\Psi$.

# Arthur packets for classical groups

- Still for $\mathbf{G} = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$, an Arthur packet $\Pi_\Psi$ defines a *local Arthur packet* for each completion $k < k_v$: A finite set $\Pi_{\Psi,v}$ of isomorphism classes of irreducible unitarizable smooth $\mathbf{G}(k_v)$-representations.
- The set of automorphic representations $\Pi_\Psi$ is then described as a certain subset (with multiplicites) of the set

$$\{\otimes_v \pi_v \ :\ \pi_v \in \Pi_{\Psi,v}\}$$

of $\mathbf{G}(\mathbb{A}_k)$-representations.

$$\text{''} \quad \Pi_\Psi = \otimes'_v \Pi_{\Psi,v} \quad \text{''}$$

# Local Arthur packets: What is it about?

- One can wonder: Does the representation theory of $p$-adic groups have a right to exist without relying on number theory?
- If your answer is positive, what is the 'true' meaning of local Arthur packets?

# Local Arthur packets: Things to notice

- Local Arthur packets may intersect, that is, an irreducible $\mathbf{G}(k_v)$-representation $\pi_v$ can belong to $\Pi_{\Psi,v} \cap \Pi_{\Psi',v}$, even when $\Pi_{\Psi,v} \neq \Pi_{\Psi',v}$.
- Yet, when $\pi_v$ is spherical, there is at most a unique local Arthur packet $\Pi$ containing it (Moeglin).
- These local packets $\Pi$ are the anti-tempered packets, that is, Aubert-dual to the tempered local Arthur packets.
- Thus, in a global Arthur packet $\Pi_\Psi = \otimes'_v \Pi_{\Psi,v}$, for all but finitely many $v$, $\Pi_{\Psi,v}$ is an anti-tempered packet.

# Size of representations of $p$-adic groups

- Now, $G$ is a reductive $p$-adic group. i.e. $G = \mathrm{Sp}_{2n}(\mathbb{Q}_p)$.
- $V_{/\mathbb{C}}$ an irreducible smooth representation $G \curvearrowright V$.
- Usually $V$ is infinite-dimensional. Still, how to quantify its 'size'?
- Maybe use compact subgroups $K < G$, because, by admissibility $\dim(V^K) < \infty$.
- Fix a basis of open compact subgroups

$$\ldots < K_{i+1} < K_i < \ldots < K_0 < G \;, \quad \cap_i K_i = \{e\}$$

- Since $V = \cup_i V^{K_i}$, $\lim_{i\to\infty} \dim(V^{K_i}) = \infty$ .

# (Vague) invariants

- The *Gelfand-Kirillov dimension* $\mathrm{GKdim}(V)$ measures the rate by which $\dim(V^{K_i})$ grows. (It can be determined by the algebraic wavefront set.)
- The *'depth'* of $V$ looks at the minimal $i$, for which $\dim(V^{K_i}) > 0$.
- What is the relation between them?
- Specifically, what are the smallest representations with respect to the two invariants?

# Minimizers

- Irreducible representations with minimal $\mathrm{GKdim}$ are a known source of interest.
- When $G$ is split, minimal 'depth' can be taken as the class of spherical representations. i.e. taking $K_0$ to be the hyperspecial maximal compact subgroup.
- Meta-claim: The two notions of small size are related.

# Example and speculations

- Each irreducible spherical representation of $GL_n(F)$ ($F$ a $p$-adic field) is the *unique* representation with minimal $\mathrm{GKdim}$ among irreducible representations on same *supercuspidal support*.

  (Follows from classical results of Moeglin–Waldspurger and Zelevinsky.)

- Speculations - Same may/should remain true for other reductive groups, with the following adjustments:
  - supercuspidal support $\rightsquigarrow$ infinitesimal character.
  - spherical $\rightsquigarrow$ weakly spherical (drop hyperspecial restriction!).
  - unique $\rightsquigarrow$ share a local Arthur packet.

# Classical groups and their Langlands reciprocity

- From now, $G$ is either $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n+1}(F)$, for a $p$-adic field $F$.
- Langlands dual group $G^\vee$ is then either $\mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$.
- Local Langlands Reciprocity: Each irreducible representation $\pi \in \mathrm{Irr}(G)$ has an *$L$-parameter*

$$\phi_\pi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$$

attached to it (up to conjugation). $W_F$ is the Weil group of the field.

# Infinitesimal characters

- The *infinitesimal character* of $\pi \in \mathrm{Irr}(G)$ is the composed homomorphism

$$\chi_\pi : W_F \xrightarrow{w \mapsto \left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right)} W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi_\pi} G^\vee ,$$

  up to conjugation.
- Terminology is motivated by an analogy to the natural notion for representations of real groups.
- For a spherical $\pi$, $\chi_\pi(\mathrm{Fr})$ is the Satake parameter classifying $\pi$.

# Basic infinitesimal characters

- Let $\mathcal{U}^\vee$ be the set of unipotent conjugacy classes in $G^\vee$.
- A class $\mathcal{O}^\vee \in \mathcal{U}^\vee$ gives by Jacobson–Morozov a homormorphism $\phi_{\mathcal{O}^\vee} : \mathrm{SL}_2(\mathbb{C}) \to G^\vee$. i.e. $\phi_{\mathcal{O}^\vee}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \in \mathcal{O}^\vee$
- We inflate $\phi_{\mathcal{O}^\vee}$ into a *basic* $L$-parameter

$$\phi_{\mathcal{O}^\vee} : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$$

by making it trivial on $W_F$.
- Representations $\pi \in \mathrm{Irr}(G)$ with $\phi_\pi = \phi_{\mathcal{O}^\vee}$ are tempered, and their infinitesimal character we denote as $\chi_{\mathcal{O}^\vee}$.

$$\mathcal{O}^\vee \quad \mapsto \quad \chi_{\mathcal{O}^\vee}$$

# Weak Arthur packets

- Barbasch–Vogan have given a local meaning to the notion of an Arthur packet for real groups. The following definition emulates their approach.
- For a unipotent conjugacy class $\mathcal{O}^\vee \in \mathcal{U}^\vee$, the set

$$\Pi^w_{\mathcal{O}^\vee} = \left\{ \pi \in \mathrm{Irr}(G) \;:\; \chi_\pi = \chi_{\mathcal{O}^\vee} \;,\; \begin{array}{l} \mathrm{GKdim}(\pi) \leq \mathrm{GKdim}(\pi'), \\ \forall \pi' \in \mathrm{Irr}(G),\; \text{s.t.} \chi_{\pi'} = \chi_{\mathcal{O}^\vee} \end{array} \right\}$$

is called a *weak Arthur packet*.

# Spherical Arthur packets revisited

- For a conjugacy class $\mathcal{O}^\vee \in \mathcal{U}^\vee$, there is a unique spherical representation $\pi \in \mathrm{Irr}(G)$ with infinitesimal character (Satake parameter) $\chi_\pi = \chi_{\mathcal{O}^\vee}$.
- Recall that a unique ('strong') local Arthur packet $\pi \in \Pi_{\mathcal{O}^\vee}$ is known to contain it.
- The packet consists of all *anti-tempered* representations that admit $\chi_{\mathcal{O}^\vee}$ as their infinitesimal character. Namely, the representations in $\Pi_{\mathcal{O}^\vee}$ are those Aubert-dual to those admitting the tempered $L$-parameter $\phi_{\mathcal{O}^\vee}$.

**Ciubotaru – Mason-Brown – Okada, 23'**

$$\Pi_{\mathcal{O}^\vee} \subset \Pi^w_{\mathcal{O}^\vee}$$

# Weak sphericity

- We say that a representation $(\pi, V) \in \mathrm{Irr}(G)$ is *weakly spherical*, when a maximal compact subgroup $K < G$ exists, so that $V^K \neq \{0\}$.
- Our groups $G$ have maximal compact subgroups that are not (conjugate to) hyperspecial:

$$K_i = G \cap \begin{pmatrix} \mathrm{GL}_i(\mathfrak{O}_F) & M_{i,N-2i}(\mathfrak{O}_F) & M_{i,i}(\mathfrak{p}_F^{-1}) \\ M_{N-2i,i}(\mathfrak{p}_F) & \mathrm{GL}_{N-2i}(\mathfrak{O}_F) & M_{N-2i,i}(\mathfrak{O}_F) \\ M_{i,i}(\mathfrak{p}_F) & M_{i,N-2i}(\mathfrak{p}_F) & \mathrm{GL}_i(\mathfrak{O}_F) \end{pmatrix} < G\,.$$

Here, $\mathfrak{p}_F < \mathfrak{O}_F < F$ is the ring of integers and its maximal ideal.

# Weakly spherical Arthur packets

- We say that a local Arthur packet $\Pi_\psi \subset \mathrm{Irr}(G)$ is *weakly spherical*, if it contains an anti-tempered weakly spherical representation $\pi \in \Pi_\psi$.
- Suspecture: Removing "anti-tempered" from the definition is harmless. (i.e when an Arthur packet contains a weakly spherical representation, it must also contain an anti-tempered weakly spherical one.)

# Main result: Symplectic case

**G. – Okada (arXiv:2404.03485)**

Let $\Pi \subset \mathrm{Irr}(\mathrm{Sp}_{2n}(F))$ be a local Arthur packet, whose constituents admit the infinitesimal character $\chi_{\mathcal{O}^\vee}$, for a unipotent conjugacy class $\mathcal{O}^\vee$ of $G^\vee$.

Then, $\Pi$ is weakly spherical, if and only if, $\Pi \subset \Pi^w_{\mathcal{O}^\vee}$.

Moreover, each weak Arthur packet $\Pi^w_{\mathcal{O}^\vee}$ is precisely the union of all weakly spherical Arthur packets whose constituents admit the infinitesimal character $\chi_{\mathcal{O}^\vee}$.

Thought process guide: One member of an Arthur packet minimizes 'depth', if and only if, all members of the packet minimize Gelfand-Kirillov dimension.

# Example

- $G = \mathrm{Sp}_8(F)$ and $\mathcal{O}^\vee$ the unipotent orbit in $\mathrm{SO}_9(\mathbb{C})$ corresponding to the partition $135$.
- Tempered $L$-parameter is

$$\phi_{\mathcal{O}^\vee} = 1 \otimes \nu_1 + 1 \otimes \nu_3 + 1 \otimes \nu_5 \ ,$$

  where $\nu_k$ is the $k$-dimensional irreducible $\mathrm{SL}_2(\mathbb{C})$-representation.
- The anti-tempered Arthur packet has $4$ representations $\Pi_{\mathcal{O}^\vee} = \{\delta, \delta', \pi, \sigma\}$.
- $\delta$ is spherical, $\delta'$ is weakly-spherical ($\pi$ is Iwahori-invariant, $\sigma$ is supercuspidal).
- While $\Pi_{\mathcal{O}^\vee}$ is the only Arthur packet containing $\delta$, there is another Arthur packet $\Pi_\psi = \{\delta', \sigma, \tau\}$ that contains $\delta'$.
- Indeed,

$$\Pi^w_{\mathcal{O}^\vee} = \Pi_{\mathcal{O}^\vee} \cup \Pi_\psi = \{\delta, \delta', \pi, \sigma, \tau\} \ .$$

# Consequences/Observations

- Weak Arthur packets are unions of Arthur packets. That was proved in parallel by Liu–Lo.
- In particular, all constituents of weak Arthur packets are unitarizable, a conjecture of Ciubotaru–Mason-Brown–Okada is settled, and a nice analogy with the real groups case is seen.
- We see a 'weak' analogue to the stated Moeglin result about uniqueness of an Arthur packet that contains a given spherical representation:
  A weak Arthur packet is the unique 'packet' with infinitesimal character $\chi_{\mathcal{O}^\vee}$ containing (anti-tempered) weakly spherical representations.

# 'Awkwardness' in the orthogonal case

- $G = \mathrm{SO}_{2n+1}(F)$ is not simply-connected, giving a central element $-I \in G^\vee$.
- Hence, $L$-parameters and their infinitesimal characters can all be tensored with the unramified quadratic character $\kappa$ of $W_F$: $\chi_{-1,\mathcal{O}^\vee}$.
- Resulting operation on $\mathrm{Irr}(G)$ is tensoring with a quadratic character $\kappa$ of $G$ (abusing notation). Has to do with the spinor norm.
- Since $\kappa$ may not be trivial on maximal compact groups (!) We say that a representation is $-1$-weakly spherical when it has non-zero $\kappa$-equivariant vectors under a maximal compact subgroup.
- More Arthur packets and weak Arthur packets need to be naturally introduced: $\Pi_{-1,\mathcal{O}^\vee}, \Pi^w_{-1,\mathcal{O}^\vee}$.

# Main theorem: Orthogonal case

**G. – Okada**

Let $\Pi \subset \mathrm{Irr}(\mathrm{SO}_{2n+1}(F))$ be a local Arthur packet, whose constituents admit the infinitesimal character $\chi_{s,\mathcal{O}^\vee}$, for a unipotent conjugacy class $\mathcal{O}^\vee$ of $G^\vee$ and $s \in \{\pm 1\}$.

Then, $\Pi$ is $(-s)$-weakly spherical, if and only if, $\Pi \subset \Pi^w_{s,\mathcal{O}^\vee}$.

Moreover, each weak Arthur packet $\Pi^w_{s,\mathcal{O}^\vee}$ is precisely the union of all $(-s)$-weakly spherical Arthur packets whose constituents admit the infinitesimal character $\chi_{s,\mathcal{O}^\vee}$.

# Three theorems

Proof of our main theorem follows from three separate results.(Though could be nice to find a direct proof!)

1. Explicit description of the Arthur packets that compose a weak Arthur packet. (extending Liu-Lo)
   - Main tool: Explicit knowledge of algebraic wavefront sets for unipotent representations (CMBO).
2. Identification of the weakly spherical spectrum in terms of the enhanced Langlands reciprocity.
   - Main tool: Kazhdan-Lusztig geometric constructions for affine Hecke algebras, and recent advances in Springer theory by Waldspurger and La.
3. Identification of Arthur packets that contain the weakly spherical spectrum.
   - Main tool: The theory of intersections of Arthur packets, as developed by Moeglin, Xu, Atobe.

# Standalone interest

**Question 1:**

Which local Arthur packets (beyond the anti-tempered packet) are included in the weak Arthur packet $\Pi^w_{\mathcal{O}^\vee}$?

**Question 2:**

Which anti-tempered representations (in $\Pi_{\mathcal{O}^\vee}$) are weakly spherical?

The answers to both questions are given in terms of the structure of the nilpotent/unipotent cone of the dual group $G^\vee$.

# Special pieces

- The conjugacy classes (or, orbits) in $\mathcal{U}^\vee$ are divided into equivalence classes known as *special pieces*.
- Each special piece consists of power-of-$2$ orbits. Toplogical order between them is the hypercube lattice.
- There is a (Barbasch–Vogan–Lusztig–Spaltenstein) duality map $d$ from $\mathcal{U}^\vee$ to the nilpotent orbits of $\mathrm{Lie}(G_{\overline{F}})$. Special pieces may be defined as the fibers of that map.
- For an orbit $\mathcal{O}^\vee \in \mathcal{U}^\vee$, we set the *relative special piece* $\mathrm{Spc}(\mathcal{O}^\vee)$ of it, to be the set of orbits in $\mathcal{U}^\vee$ that share a special piece with $\mathcal{O}^\vee$ and are contained in its closure.
- Still $|\mathrm{Spc}(\mathcal{O}^\vee)| = 2^k$.

# Weakly spherical Arthur parameters

- One meaningful invariant of Arthur parameters $\psi \in \Psi(G)$ is their $\mathrm{SL}_2$-type:
- Namely, when viewed as a homomorphism

$$\psi : W_F \times \mathrm{SL}_2^L(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to G^\vee ,$$

this is the restriction $\psi|_{\mathrm{SL}_2^A(\mathbb{C})}$, whose isomorphism class is again parameterized by an orbit $\mathcal{O}_\psi^\vee \in \mathcal{U}^\vee$.

**Proposition**

For any $\mathcal{O}_1^\vee \in \mathcal{U}^\vee$ and any $\mathcal{O}_2^\vee \in \mathrm{Spc}(\mathcal{O}_1^\vee)$, there is a unique Arthur parameter $\psi = \psi_{\mathcal{O}_1^\vee, \mathcal{O}_2^\vee} \in \Psi(G)$ with infinitesimal character coming from $\mathcal{O}_1^\vee$ and $\mathrm{SL}_2$-type $\mathcal{O}_2^\vee$.

$$\chi_\psi = \chi_{\mathcal{O}_1^\vee} , \quad \mathcal{O}_\psi^\vee = \mathcal{O}_2^\vee .$$

For example, $\Pi_{\psi_{\mathcal{O}^\vee, \mathcal{O}^\vee}} = \Pi_{\mathcal{O}^\vee}$ is the anti-tempered Arthur packet.

# Weakly spherical Arthur packets

## Theorem (G.-Okada)

For any unipotent orbit $\mathcal{O}_1^\vee \in \mathcal{U}^\vee$, the weak Arthur packet attached to it is decomposed as a (non-disjoint) union of $|\mathrm{Spc}(\mathcal{O}_1^\vee)|$ Arthur packets

$$\Pi^w_{\mathcal{O}_1^\vee} = \bigcup_{\mathcal{O}_2^\vee \in \mathrm{Spc}(\mathcal{O}_1^\vee)} \Pi_{\psi_{\mathcal{O}_1^\vee, \mathcal{O}_2^\vee}} .$$

**Question 2:**

Which anti-tempered representations (in $\Pi_{\mathcal{O}^\vee}$) are weakly spherical?

# A side plot: Representation theory of Weyl groups

- Our groups $G^\vee$ (and $G$...) have the finite group $W_n$ of signed permutations as their Weyl group.
- For each unipotent conjugacy class $\mathcal{O}^\vee$ in $G^\vee$, Springer theory constructs an action of $W_n$ on the cohomology space $H^*(\mathcal{B}_{\mathcal{O}^\vee})$ of the variety of Borel subgroups of $G^\vee$ containing a fixed representative of $u \in \mathcal{O}^\vee$.
- The component (2-)group $A(\mathcal{O}^\vee) := Z_{G^\vee}(u)/Z_{G^\vee}(u)^\circ$ acts on $H^*(\mathcal{B}_{\mathcal{O}^\vee})$ as well, commuting with the $W_n$-action.
- Each irreducible local system $\rho \in \widehat{A(\mathcal{O}^\vee)}$ gives a $W_n$-representation

$$\Sigma(\mathcal{O}^\vee, \rho) = \mathrm{Hom}_{A(\mathcal{O}^\vee)}(\rho, H^*(\mathcal{B}_{\mathcal{O}^\vee}))\ .$$

# Kazhdan–Lusztig $K$-theory construction

- For the principal block of representations of split $p$-adic groups, Kazhdan–Lusztig have adopted a Springer-like approach to construct Langlands reciprocity.
- Idea is that these cases are equivalent to representation of an *affine Hecke algebra*, which is viewed as a *quantized* version of the *affine* Weyl group (of the $p$-adic group in question).
- Lusztig later extended this approach to treat all *unipotent* representations.
- Bottom line for our needs: A geometric construction and parameterization

$$\Pi_{\mathcal{O}^\vee} = \left\{ \delta(\mathcal{O}^\vee, \rho) \;:\; \rho \in \widehat{A(\mathcal{O}^\vee)}_0 \right\}$$

of anti-tempered representations is in place.

# Weak sphericity translated to Springer theory

- Lusztig and Reeder showed that when "$q \to 1$" ($q =$ residue characteristic of $F$) is suitably performed, one obtains the $W_n$-representation $\Sigma(\mathcal{O}^\vee, \rho)$ out of the $G$-representation $\delta = \delta(\mathcal{O}^\vee, \rho)$, whenever $\delta$ is Iwahori-invariant.
- Turns out weak-sphericity can be detected on the "$q \to 1$" level!

**Proposition**

$$\dim\left(\Sigma(\mathcal{O}^\vee, \rho)^{W_i \times W_{n-i}}\right) = \dim\left(\delta(\mathcal{O}^\vee, \rho)^{K_i}\right)$$

Here, $W_i \times W_{n-i} < W_n$ is the (non-parabolic, in Coxeter formalism) subgroup of signed permutations perserving $\{1, \ldots, i\}$.

# Green theory

- Want to know when is $\Sigma(\mathcal{O}^\vee, \rho)^{W_i \times W_{n-i}} \neq \{0\}$.
- Recall, $\Sigma(\mathcal{O}^\vee, \rho)$ is not irreducible. Determining its decomposition to irreducible representations has to do with the theme of *Green functions*.
- Recently, Waldspurger devised closed formulas for such a decomposition for the $G^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$ case. Methods were extended to $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$ by La.

# Lusztig's canonical quotient

**Theorem: Symplectic case (G.–Okada)**

The representation $\delta(\mathcal{O}^\vee, \rho)$ is weakly spherical, if and only if, the character $\rho$ of the component group $A(\mathcal{O}^\vee)$ factor through Lusztig's canonical quotient.

- The subgroup $A^\dagger(\mathcal{O}^\vee) < \widehat{A(\mathcal{O}^\vee)}$ of characters factoring through that quotient has its own history.
- Recall that each non-zero $\Sigma(\mathcal{O}^\vee, \rho)$ has an *irreducible* summand $L(\mathcal{O}^\vee, \rho) = \mathrm{Hom}_{A(\mathcal{O}^\vee)}(\rho, H^{top}(\mathcal{B}_{\mathcal{O}^\vee})) \in \mathrm{Irr}(W_n)$.
- $\mathrm{Irr}(W_n)$ is divided into *Kazhdan–Lusztig (two-sided) cells*. Those are conveniently in bijection with the set of special pieces in $\mathcal{U}^\vee$.
- Achar–Sage: $A^\dagger(\mathcal{O}^\vee)$ consists of those characters $\rho$, for which the KL cell of $L(\mathcal{O}^\vee, \rho)$ matches the special piece of $\mathcal{O}^\vee$.

# Thank you for listening!