Restricting real group reps to K

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Outline

Old ideas from old mathematicians

New ideas, also from old mathematicians

The conjecture from the abstract

Old setting

Compact groups K are relatively easy...

Noncompact groups reductive G are relatively hard.

Harish-Chandra *et al.* idea:

$$\text{understand } \pi \in \widehat{G} \leftrightsquigarrow \text{understand } \pi|_K$$

(nice compact subgroup $K \subset G$).

In order to pursue this idea, need to describe $\widehat{K}$.

Fix max torus $T_{c,0} \subset K_0$; set $T_c = K^{T_{c,0}}$ Cartan subgp of K.

Old: Cartan $T_c \subset K \rightsquigarrow \mu \in \widehat{K} \rightsquigarrow$ extremal weights

$$\text{HW}(\mu) = \{\tau \in X^*(T_c)\}, \quad \text{single } W(K, T_c) \text{ orbit.}$$

Partially order $\widehat{K}$ by size of extremal weights.

$\pi \in \widehat{G} \rightsquigarrow$ finite set $\text{LKTs}(\pi) \subset \widehat{K}$, lowest K-types of π.

Lowest K-types for $Sp(4,\mathbb{R})$

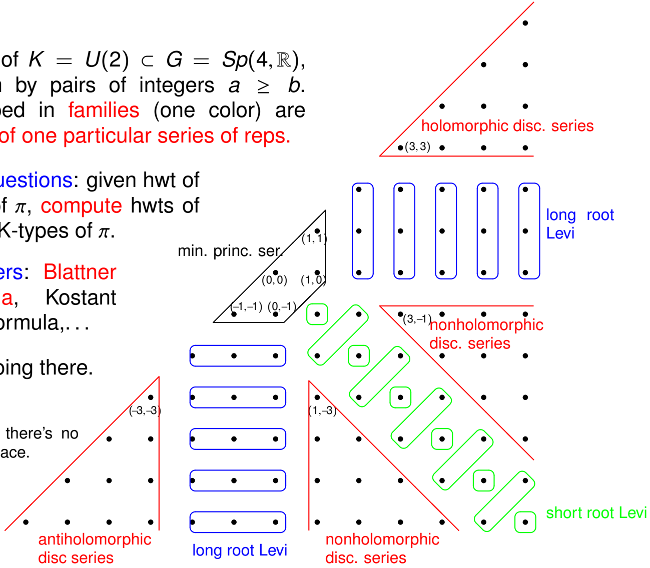
Reps of $K = U(2) \subset G = Sp(4,\mathbb{R})$, param by pairs of integers $a \geq b$. Grouped in families (one color) are LKTs of one particular series of reps.

Old questions: given hwt of LKT of π, compute hwts of other K-types of π.

Answers: Blattner formula, Kostant mult formula,. . .

Not going there.

Anyway there's no more space.



What's wrong with this picture?

What does order $\widehat{K}$ by size of extremal weights mean?

Need to pick inner product on torus chars $X^*(T_c)$.

This is possible, but not entirely natural.

Two LKTs of same π can have different $\|$extr wt$\|$.

$Sp(4,\mathbb{R})$: $(2,0)$, $(2,2)$ (first blue box) are LKTs of one π.

Extr wt of a disc ser LKT may not be minimal among extr wts of all its K-types.

Extr wts of LKTs(π) are not easily visible in Langlands param(π).

Old way to fix these problems: if K-type μ has extr wt τ, pick pos roots for K annihilating τ weight; add them up, getting $2\rho_K \in X^*(T_c)$. Set

$$\|\mu\|_{\text{old}} = \text{length of } (\tau + 2\rho_K).$$

This fixes some of the problems above.

But it's better to cast off the shackles of highest weight theory. . . .

Harish-Chandra's discrete series

Recall max torus $T_c \subset K \subset G$ real reductive.

Problems above: $\leftrightsquigarrow$ extremal weight of LKT of disc series rep is close to but not equal to HC param of disc series.

Theorem (Harish-Chandra) G has disc series reps if and only if T_c is a max torus in G. In this case disc series reps are in one-to-one correspondence with regular $W(K, T_c)$ orbits on the shifted character lattice $X^*(T_c) + \rho$.

$\lambda \in X^*(T_c) + \rho$ regular if $\lambda(\alpha^\vee) \neq 0$, all coroots $\alpha^\vee \in \mathfrak{t}_c$ of T_c in G.

$$X^*_{\text{reg}}(T_c) = \{\text{shifted reg chars}\}, \qquad \pi(\lambda) = \text{disc series rep}.$$

Theorem (Hecht-Schmid) Suppose $\lambda \in X^*_{\text{reg}}(T_c)$. Write ρ for the half sum of G-roots whose coroots are positive on λ, and ρ_K for the half sum of K-roots. Then the unique LKT μ of $\pi(\lambda)$ has extremal weight

$$\mu = \lambda + \rho - 2\rho_K.$$

Definition Suppose $\lambda \in i\mathfrak{t}_c^*$. The height of λ is

$$\text{ht}(\lambda) = \Big(\sum_{\text{coroots } \alpha^\vee} |\lambda(\alpha^\vee)|\Big)/2.$$

The height of a discrete series $\pi(\lambda)$ is $\text{ht}(\lambda)$.

Replacing Cartan-Weyl by Harish-Chandra

Summary of last page:

1. $T_c \subset K$ maximal torus, $X^*_{\text{reg}}(T_c)$ shifted by ρ regular characters.
2. if μ is LKT of discrete series $\pi(\lambda)$, then μ has extremal weight $\lambda + \rho - 2\rho_K$.
3. $\text{ht}(\lambda) =$ sum of pos coroot values on λ.
4. ht takes nonneg integer values on $X^*(T_c) + \rho$ if G linear (rational in general).

If μ is LKT of disc ser $\pi(\lambda)$ (with $\lambda \in X^*_{\text{reg}}(T_c)$) then write

$$\text{LAM}(\mu) = \{\lambda \in X^*_{\text{reg}}(T_c) \mid \mu = \text{LKT}(\pi(\lambda)\},$$

the lambda parameter of μ: single $W(K, T_c)$ orbit.

Lambda parameter of $\mu \in \widehat{K}$ is Harish-Chandra (= Langlands) parameter of discrete series having μ as lowest K-type.

What about other reps of K?

Suppose $H = TA$ max torus in G (T cpt, A vec group). Then

$$G^A = MA, \quad M \text{ reductive} \supset M \cap K \text{ max cpt} \supset T \text{ Cartan of } M \cap K$$

MA is Levi subgroup of cuspidal parabolic $P = MAN$.

Characters of H are lattice times cplx vec space

$$X^*(H) = X^*(T) \times X^*(A) = X^*(T) \times \mathfrak{a}^*_{\mathbb{C}}$$

HC: Disc series reps for M indexed by $W(M, T)$ orbits of shifted M-reg chars $\lambda \in X^*(T) + \rho_M$.

Shifted M-reg char $\gamma = (\lambda, \nu) \in X^*_{\text{reg}}(H) \rightsquigarrow$ relative disc series

$$\pi_{MA}(\gamma) = \pi_M(\lambda) \otimes \nu \in \widehat{MA}.$$

Principal series rep of G is $\pi_G(\gamma) = \mathrm{Ind}_{MAN}^G(\pi_{MA}(\gamma) \otimes 1)$.

If μ is LKT of princ ser $\pi(\gamma)$ (with $\gamma|_T = \lambda \in X^*_{\text{reg}}(T)$) then write

$$\text{LAM}(\mu) = \{\lambda \in X^*_{\text{reg}}(T) \mid \mu \in \text{LKTs}(\pi(\gamma), \lambda = \gamma|_T\},$$

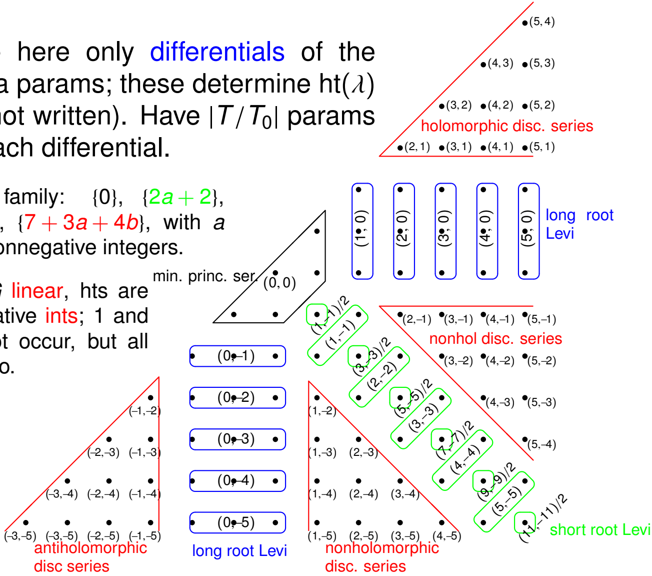
the lambda parameter of μ: single $W(G, T)$ orbit.

Lambda parameters for $Sp(4, \mathbb{R})$

I write here only differentials of the lambda params; these determine $\mathrm{ht}(\lambda)$ (also not written). Have $|T/T_0|$ params with each differential.

hts by family: $\{0\}$, $\{2a+2\}$, $\{3b+3\}$, $\{7+3a+4b\}$, with a and b nonnegative integers.

Since G linear, hts are nonnegative ints; 1 and 5 do not occur, but all others do.



Parametrizing $\widehat{K}$ by λs

Theorem Each $\mu \in \widehat{K}$ is a lowest K-type of a principal series rep $\pi_G(\gamma)$, with $H = TA$ a Cartan in G and $\gamma = (\lambda, \nu) \in X^*_{\text{reg}}(H)$ a shifted regular character. Pair (T, λ) is uniquely determined by μ up to conjugation by $W(G, H)$.

If G linear, thm is restr-to-K version of Langlands classification for G. Parameter λ is specified up to $W_{\mathbb{C}}(G, T)$ by the restriction of the Langlands parameter to the maximal compact subgroup of the Weil group.

$W_{\mathbb{C}}(G, T)$ means subgroup of $W(G_{\mathbb{C}}, T_{\mathbb{C}})$ preserving T.

Version of thm with lim disc series indexes $\widehat{K}$ bijectively.

Much works for p-adic G. Big difference is that first statement of the theorem is false in the p-adic case: most irreducibles of K are not lowest K-types of principal series.

Example of $SL(3,R)$

$G = SL(3,R)$, $K = SO(3) \supset T_c = SO(2)$, $X^*(T_c) = \mathbb{Z}$.

Here is a picture of $\widehat{K}$ and some of the numbers we've attached.

(dim, ht)	(1, 0)	(3, 0)	(5, 2)	(7, 4)	(9, 6)	(11, 8)
lambda	(0, triv)	(0, nontriv)	1	2	3	4
	●	●	●	●	●	●
highest wt	0	1	2	3	4	5
mult in sph ps	1	0	2	1	3	2
mult in nonsph	0	1	1	2	2	3
mult in fund #1	0	0	1	1	2	2

G has two conj classes of Cartan subgps:

1. $H_c = G^T = T_c A_c \simeq \mathbb{C}^\times$ fundamental; $T_c \simeq SO(2)$
2. $H_s = T_s A_s =$ diag subgp $\simeq (\mathbb{R}^\times)^2$ split; $T_s \simeq \{\pm 1\}^2$

K-types of sph princ ser have dims $1, 5, 7, 9, 11 \ldots$, hts $0, 2, 4, 6 \ldots$.

K-types of nonsph princ ser have dims $3, 5, 7, 9, 11 \ldots$, hts $0, 2, 4, 6 \ldots$.

K-types of mth fund ser have dims $2m+3, 2m+5, 2m+7, 2m+9 \ldots$, hts $2m, 2m+2, 2m+4, 2m+6, \ldots$.

So what is this toy conjecture?

To relate lambda params to branching for G, set (for $\pi \in \widehat{G}$)

$$\text{LAMs}(\pi) := \{\text{LAM}(\tau) \mid \tau \in \widehat{K} \text{ occurring in } \pi\}.$$

$$\text{hts}(\pi) := \{\text{ht}(\tau) \mid \tau \in \widehat{K} \text{ occurring in } \pi\}.$$

Branching to K for (HC's generalized) principal series reps is a classical problem, so for any lambda parameter ψ, put

$$\text{LAMs}(\psi) := \text{LAMs}(\pi(\psi, \nu)), \qquad \text{hts}(\psi) := \text{hts}(\pi(\psi, \nu));$$

the principal series restriction $(\pi(\psi, \nu))|_K$ does not depend on ν.

Suppose that $H_q = T_q A_q$ is a quasisplit Cartan (that is, contained in a real Borel subgroup) for a quasisplit group G_q.

Conjecture. Suppose that λ_1 and λ_2 are (shifted M-regular) characters of T_q, of height zero, having the same restriction to $Z(G) \cap K \subset T_q$. Then $\text{hts}(\lambda_1) = \text{hts}(\lambda_2)$.

If G is split, then T_q is $(\pm 1)^{\text{rank}(G)}$: there are $2^{\text{rank}(G)}$ possible λ. When two λs are in different W orbits, restrictions to K of the principal series are different: e.g. sets of lowest K-types are disjoint. Conj says sets of heights of the K-types are nevertheless often exactly the same.

Some evidence

$SL(2,\mathbb{R})$ has two kinds principal series restrictions to K:

spherical $= \{$ even $SO(2)$ wts $\}$, nonspherical $= \{$ odd wts $\}$.

So sets of heights are $[0,1,3,5,7,\ldots]$ and $[0,2,4,6,\ldots]$: different. But sph $\neq$ nonsph on $Z_K(G)$: so conj is true.

$SL(3,\mathbb{R})$ also has two kinds of principal series restrictions to K: spherical $\supset$ (all $SO(3)$ irrs except 3-diml), and nonspherical $\supset$ (all $SO(3)$ irrs except 1-diml. Since 1- and 3-diml reps both have height 0, sets of heights agree, as conjecture requires.

E_8 has three kinds of principal series restrictions to K, with lowest K-types of dims 1, 120, and 135. For each of these, the set of heights of K-types up to 100 is [0,29,46,57,58,68,75,84,87,91,92]. I have not computed further.