Enhanced Langlands parameters and Hecke algebras

Anne-Marie Aubert

Institut de Mathématiques de Jussieu – Paris Rive Gauche
C.N.R.S., Sorbonne Université and Université Paris Cité

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Notation

- G group of F-rational points of a connected reductive algebraic F-group, with F a non-archimedean local field (finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$). We will refer to G as a p-adic group.
- W_F Weil group of F
- $G^\vee$ complex reductive group with root datum dual to that of G
- ${}^LG := G^\vee \rtimes W_F$ the L-group of G

The Local Langlands Correspondence (LLC)

predicts a surjective map, satisfying several properties,

$$\left\{\begin{array}{c}\text{irred. smooth}\\ \text{repres. } \pi \text{ of } G\end{array}\right\}/\text{iso.} \xrightarrow{\mathcal{L}} \left\{\begin{array}{c} L\text{-parameters} \\ \text{i.e. cont. homomorphisms} \\ \varphi_\pi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG \end{array}\right\}/G^\vee\text{-conj.}$$

with finite fibers, called L-packets.

Remarks

- In order to obtain a bijection LLC between the group side and the Galois side, the conjectural map $\mathcal{L}$ was later enhanced: on the Galois side, one considers enhanced L-parameters: (φ_π, ρ_π), where the enhancement ρ_π is a representation of a certain component group.
- It may be useful to consider simultaneously inner twists of a given group G. This leads to "compound" L-packets.
- There are several ways one can enhance L-parameters in order to capture information about the internal structure of L-packets. For the most part, the choice of enhancement has to do with the type of inner twist of G we consider.

A bijective LLC

has been constructed in particular in the following cases:

- $G = F^\times = \mathrm{GL}_1(F)$ Class field theory (first half of the 20th century);
- $G = \mathrm{GL}_n(F)$ Laumon-Rapoport-Stuhler (1993) $\mathrm{char}(F) > 0$, Harris-Taylor (1998), Henniart (2000), Scholze (2010);
- $G = \mathrm{SL}_n(F)$ (and its inner twists) Hiraga-Saito (2012) $\mathrm{char}(F) = 0$; A.-Baum-Plymen-Solleveld (2016) $\mathrm{char}(F) > 0$;
- $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$ ($\mathrm{char}(F) = 0$) Arthur (2013);
- $G = \mathrm{G}_2(F)$ A.-Xu (2022), Gan-Savin (2022);
- for all the unipotent representation of an arbitrary p-adic group G [Lusztig (1995 & 2002), Feng, Opdam, Solleveld (2020-2022)].

The Bernstein decomposition [Bernstein, 1984]

The category $\mathfrak{R}(G)$ of smooth representations of a p-adic group G is a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G) \tag{1}$$

of the full subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, where

- $\mathfrak{B}(G) = \{\mathfrak{s} = (L, \mathfrak{X}_{\mathrm{nr}}(L)\cdot\sigma)_G\}$. Notation $\mathfrak{s} = [L,\sigma]_G$.
  - L Levi subgroup of G and σ supercuspidal smooth irrep of L
  - $\mathfrak{X}_{\mathrm{nr}}(L)$ group of unramified characters of L
- $\mathfrak{R}^{\mathfrak{s}}(G)$ subcategory of $\mathfrak{R}(G)$ whose objects are the representations π such that every irreducible G-subquotient of π has its supercuspidal support in $\mathfrak{s}$.

Example: The irred. objects of $\mathfrak{R}^{\mathfrak{s}_1}(G)$, where $\mathfrak{s}_1 = [T, \mathrm{triv}]_G$, are the Iwahori-spherical irreps. of G.

An extended finite Weyl group

Set $W_{\mathfrak{s}} := \mathrm{N}_G(\mathfrak{s})/L$. It is an extended finite Weyl group:

$$W_{\mathfrak{s}} = W_{\mathfrak{s}}^{\circ} \rtimes \Gamma_{\mathfrak{s}} \tag{2}$$

where $W_{\mathfrak{s}}^{\circ}$ is the finite Weyl group, with root system $\Sigma_{\mathfrak{s}}$, the set of roots for which the associated Harish-Chandra μ-function has a zero on $\mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma$, and $\Gamma_{\mathfrak{s}}$ is the stabilizer of the set of positive roots.

A root datum attached to $\mathfrak{s}$

Set $\mathfrak{X}_{\mathrm{nr}}(L, \sigma) := \{\chi \in \mathfrak{X}_{\mathrm{nr}}(L) : \sigma \otimes \chi \cong \sigma\}$ and $\mathcal{L}_{\sigma} := \bigcap_{\chi \in \mathfrak{X}_{\mathrm{nr}}(L,\sigma)} \ker \chi$.
Let L_1 be the subgroup of L generated by all compact subgroups of L.
Let $\alpha \in \Sigma_{\mathfrak{s}}$ and $h_{\alpha}^{\vee}$ the unique generator of $(\mathcal{L}_{\sigma} \cap L_1)/L_1 \cong \mathbb{Z}$ such that $|\alpha(h_{\alpha}^{\vee})|_F > 1$. We write $R_{\mathfrak{s}} := \{h_{\alpha}^{\vee} : \alpha \in \Sigma_{\mathfrak{s}}\}$.
Then $T_{\mathfrak{s}} := \mathfrak{X}_{\mathrm{nr}}(L)/\mathfrak{X}_{\mathrm{nr}}(L, \sigma)$ is a complex torus and

$$\mathfrak{R}_{\mathfrak{s}} := (X^*(T_{\mathfrak{s}}), R_{\mathfrak{s}}, X_*(T_{\mathfrak{s}}), R_{\mathfrak{s}}^{\vee})$$

is a root datum.

Weights functions

These are functions $\lambda, \lambda^* \colon R_{\mathfrak{s}} \to \mathbb{R}_{\geq 0}$, such that

- if $\alpha, \beta \in R_{\mathfrak{s}}$ are $W_{\mathfrak{s}}^{\circ}$-associate, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,
- if $\alpha^\vee \notin 2X_*(T_{\mathfrak{s}})$, then $\lambda^*(\alpha) = \lambda(\alpha)$. (It is always the case except possibly for short roots α in a type B component of $R_{\mathfrak{s}}$.)

They are defined by

$$\lambda(h_\alpha^\vee) := \log(q_\alpha q_{\alpha^*})/\log(q) \quad \text{and} \quad \lambda^*(h_\alpha^\vee) := \log(q_\alpha q_{\alpha^*}^{-1})/\log(q_\alpha),$$

where $q_\alpha, q_{\alpha^*} \in \mathbb{R}_{\geq 1}$ come from Silberger's computation of the Harish-Chandra μ-function associated to α.

Definition

The affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^* \, q^{1/2})$ is the vector space $\mathcal{H}(W_{\mathfrak{s}}^{\circ}, q^{\lambda(\alpha)}) \otimes_{\mathbb{C}} \mathbb{C}[X^*(T_{\mathfrak{s}})]$ with the with the multiplication rules:

- $\mathcal{H}(W_{\mathfrak{s}}^{\circ}, q^{\lambda(\alpha)})$ and $\mathbb{C}[X^*(T_{\mathfrak{s}})]$ are embedded as subalgebras;
- for $\alpha \in \Delta_{\mathfrak{s}}$ (a basis of $R_{\mathfrak{s}}$) and $x \in X^*(T_{\mathfrak{s}})$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} =$$

$$\Big((q^{\lambda(\alpha)} - 1) + \theta_{-\alpha}(q^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - q^{(\lambda(\alpha)-\lambda^*(\alpha))/2})\Big) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}},$$

  where $\{\theta_x \,:\, x \in X\}$ is a basis of $\mathbb{C}[X^*(T_{\mathfrak{s}})]$.

It is an associative algebra with unit element $T_1 \otimes \theta_0$.

Structure of blocks [Heiermann, Solleveld]

In many cases, it is known that

$$\mathfrak{R}^{\mathfrak{s}}(G) \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{H}(G, \mathfrak{s})) \tag{3}$$

where $\mathcal{H}(G, \mathfrak{s})$ is a (twisted) extended affine Hecke algebra:

$$\mathcal{H}(G, \mathfrak{s}) = \mathcal{H}(G, \mathfrak{s})^{\circ} \rtimes \mathbb{C}[\Gamma_{\mathfrak{s}}, \natural_{\mathfrak{s}}], \tag{4}$$

and $\mathcal{H}(G, \mathfrak{s})^{\circ} = \mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^* \, q^{1/2})$.

Remark

For instance (3) is satisfied if the restriction of σ to L_1 is multiplicity free. It is the case, in particular, when the maximal F-split central torus of L has dimension ≤ 1, and also when L is quasi-split and σ is generic.

A general strategy to construct the LLC (A.-Moussaoui-Solleveld):

1. Define an analogue of Bernstein's decomposition on the Galois side of the correspondence.
2. Attach a (twisted) extended affine Hecke algebra to each "Galois block".
3. Construct an explicit LLC for p-adic groups "block by block" via a correspondence between Hecke algebras: prove that the (twisted) extended affine Hecke algebras on each side of the correspondence are isomorphic, or at least closely related: in particular, we need that their simple modules are in bijection.

Theorem [A.-Moussaoui-Solleveld, 2023]

The above strategy works for all pure inner forms of quasi-split p-adic classical groups (symplectic, (special) orthogonal, general (s)pin, and unitary groups) and the obtained correspondence coincides with Arthur's LLC.

Definition

For simplicity, suppose G pure inner twist of a quasi-split group. We set

$$S_\varphi := \mathrm{Z}_{G^\vee}(\varphi(W_F')). \tag{5}$$

An enhanced L-parameter is a pair (φ, ρ) where φ is an L-parameter for G and $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, with $\mathcal{S}_\varphi := S_\varphi / S_\varphi^\circ$.
For φ a given L-parameter, ρ is called an enhancement of φ.

Action of $G^\vee$ on the set of enhanced L-parameters:

$$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho), \quad \text{for } g \in G^\vee,$$

where ${}^g\rho$: $h \mapsto \rho(g^{-1}hg)$.
Φ_{e}: set of $G^\vee$-conjugacy classes of enhanced L-parameters.
$\Phi_{\mathrm{e}}(G)$: set of $G^\vee$-conjugacy classes of G-relevant enhanced L-parameters.

Definitions

- $\mathcal{G}_\varphi := \mathrm{Z}_{G^\vee}(\varphi(W_F))$: a (possibly disconnected) complex reductive group
- $u = u_\varphi := \varphi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$: unipotent element of $\mathcal{G}_\varphi$
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u))$.

We have

$$\mathcal{S}_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi). \tag{6}$$

Main idea:
(6) will allow us to use the generalized Springer correspondence for the complex group $\mathcal{G}_\varphi$ in order to understand the structure of the L-packets for the p-adic group G.

Generalized Springer variety [Lusztig, Invent. math. 1984]

Let $\mathcal{G}$ be a connected reductive group over $\mathbb{C}$, and let

- $\mathcal{P} = \mathcal{L}\mathcal{U}$ parabolic subgroup of $\mathcal{G}$
- $u \in \mathcal{G}$ and $v \in \mathcal{L}$ unipotent elements.

The group $\mathrm{Z}_{\mathcal{G}}(u) \times \mathrm{Z}_{\mathcal{L}}(v)\mathcal{U}$ acts on the variety

$$Y_{u,v} := \left\{ y \in \mathcal{G} \,:\, y^{-1}uy \in v\mathcal{U} \right\}$$

by $(g, p) \cdot y := gyp^{-1}$, with $g \in \mathrm{Z}_{\mathcal{G}}(u)$, $p \in \mathrm{Z}_{\mathcal{L}}(v)\mathcal{U}$ and $y \in Y_{u,v}$.

The group $A_{\mathcal{G}}(u) \times A_{\mathcal{L}}(v)$ acts on the set of irreducible components of $Y_{u,v}$ of maximal dimension (i.e. $\dim \mathcal{U} + \frac{1}{2}(\dim \mathrm{Z}_{\mathcal{G}}(u) + \dim \mathrm{Z}_{\mathcal{L}}(v))$). Let $\sigma_{u,v}$ denote the corresponding permutation representation.

Definition [Lusztig, Invent. math. 1984]

Let $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$. Then ρ is called cuspidal if

$$\langle \rho, \sigma_{u,v} \rangle_{A_{\mathcal{G}}(u)} \neq 0 \text{ for any unipotent } v \in \mathcal{L} \quad \Rightarrow \quad \mathcal{P} = \mathcal{G},$$

where $\langle\ ,\ \rangle_{A_{\mathcal{G}}(u)}$ is the usual scalar product on the space of class functions on $A_{\mathcal{G}}(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

Note: If (u, ρ) is cuspidal, then $\mathcal{C}$ is a distinguished (i.e. $\mathcal{C}$ does not meet the unipotent variety of $\mathcal{L}$ for any $\mathcal{L} \neq \mathcal{G}$). However, in general not every distinguished unipotent class supports a cuspidal representation.

Theorem [Lusztig, *loc. cit.*]

Let $\mathcal{C}$ be a unipotent class in $\mathcal{G}$ and $\mathcal{E}$ an irreducible $\mathcal{G}$-equivariant local system on $\mathcal{C}$. The IC-sheaf $\mathcal{F}_\rho := \mathrm{IC}(\mathcal{C}, \mathcal{E}_\rho)$ occurs as a summand of $\mathrm{i}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\mathrm{IC}(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, for some triple $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, where $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$ with Levi subgroup $\mathcal{L}$ and $(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}})$ is a cuspidal unipotent pair in $\mathcal{L}$. Moreover, the triple $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$ is unique up to $\mathcal{G}$-conjugation.

Definition

Let $\rho \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$. The cuspidal support of (u, ρ), denoted by $\mathrm{Sc}^{\mathcal{G}^\circ}(u, \rho)$, is defined to be

$$(\mathcal{L}, (v, \rho_{\mathrm{cusp}}))_{\mathcal{G}}, \quad \text{where } v \in \mathcal{C}_{\mathrm{cusp}} \text{ and } \rho_{\mathrm{cusp}} \leftrightarrow \mathcal{E}_{\mathrm{cusp}}. \tag{7}$$

Disconnected complex reductive groups [A.-Moussaoui-Solleveld, 2018]

Let $\mathcal{G}$ be a possibly disconnected reductive group over $\mathbb{C}$, with identity component $\mathcal{G}^\circ$. Let $u \in \mathrm{U}(\mathcal{G})$ and $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$. We observe that $A_{\mathcal{G}^\circ}(u) \subset A_{\mathcal{G}}(u)$.

- The pair (u, ρ) is called cuspidal if the restriction of ρ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ρ° such that one (or equivalently any) of the pairs (u, ρ°) is cuspidal.
- We set $\mathcal{T} := \mathrm{Z}_{\mathcal{L}}^\circ$ and $\mathcal{M} := \mathrm{Z}_{\mathcal{G}}(\mathcal{T})$. The cuspidal support of (u, ρ) is a (well-defined) triple $(\mathcal{M}, v, \rho_{\mathrm{cusp}})_{\mathcal{G}}$, where $\rho_{\mathrm{cusp}}^\circ$ occurs in the restriction of ρ_{cusp} to $A_{\mathcal{G}^\circ}(u)$.

Remark

By the Jacobson–Morozov theorem, any unipotent element v of $\mathcal{L}$ can be extended (in a unique way up to $\mathrm{Z}_{\mathcal{L}}(v)^{\circ}$-conjugation) to a homomorphism of algebraic groups

$$j_v \colon \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L} \text{ satisfying } j_v \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = v. \tag{8}$$

Definition [A.-Moussaoui-Solleveld, 2018]

An enhanced L-parameter $(\varphi, \rho) \in \Phi_{\mathrm{e}}$ is called cuspidal if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$),
- (u_φ, ρ) is a *cuspidal pair* in $\mathcal{G}_\phi$.

We denote by $\Phi_{\mathrm{e,cusp}}(G)$ the set of $G^\vee$-conjugacy of cuspidal enhanced L-parameters for G.

The generalized Springer correspondence allows us to define a cuspidal support map

$$\mathrm{Sc}\colon \Phi_{\mathrm{e}}(G) \to \bigcup_{L \text{ Levi de } G} \Phi_{\mathrm{e,cusp}}(L). \tag{9}$$

Definition of the map Sc

Let $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$. We define $\varphi_v\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Z}_{G^\vee}(\mathcal{T})$ by

$$\varphi_v(w,x) := \varphi(w,1)\cdot \chi_{\varphi,v}(\|w\|^{1/2}) \cdot j_v(x) \quad \text{for all } w \in W_F,\ x \in \mathrm{SL}_2(\mathbb{C})$$

where

$$\chi_{\varphi,v}\colon z \mapsto \varphi\left(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1}\end{smallmatrix}\right)\right)\cdot j_v\left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z\end{smallmatrix}\right) \qquad \text{for } z \in \mathbb{C}^\times.$$

The cuspidal support of (φ,ρ) is defined to be

$$\mathrm{Sc}(\varphi,\rho) := (\mathrm{Z}_{G^\vee}(\mathcal{T}), (\varphi_v, \rho_{\mathrm{cusp}})). \tag{10}$$

Cuspidality Conjecture [A-Moussaoui-Solleveld, 2018]

The cuspidal G-relevant enhanced Langlands parameters correspond by the LLC to the irreducible supercuspidal representations of G:

$$\mathrm{LLC}\colon \mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(G). \tag{11}$$

State of art

The cuspidality conjecture is known to hold for all the Levi subgroups (including the groups themselves) of

- general linear groups and split classical p-adic groups [Moussaoui, 2017],
- inner forms of linear groups and of special linear groups, and quasi-split unitary p-adic groups [A-Moussaoui-Solleveld, 2018],
- the p-adic group G_2 [A-Xu, 2022],
- pure inner forms of quasi-split classical p-adic groups [A-Moussaoui-Solleveld, 2022].

Definition

- $L^\vee$ Langlands dual group of a Levi subgroup L
- $\mathfrak{X}_{\mathrm{nr}}(L^\vee) := \{\zeta\colon W_F/I_F \to \mathrm{Z}^\circ_{L^\vee}\}$, which acts on the set of cuspidal enhanced L-parameters for L.
- $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{G^\vee}$ the $G^\vee$-conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{\mathrm{nr}}(L^\vee)\cdot(\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}}))$, where $(\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}}) \in \Phi_{\mathrm{e,cusp}}(L)$
- $\mathfrak{B}^\vee(G)$ the set of such $\mathfrak{s}^\vee$.
- $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$: fiber of $\mathfrak{s}^\vee$ under the map Sc.

Theorem [A.-Moussaoui-Solleveld, 2018]

The set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of enhanced L-parameters is partitioned into *series à la Bernstein* as

$$\Phi_{\mathrm{e}}(G) = \coprod_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi^{\mathfrak{s}^\vee}(G). \tag{12}$$

A variant of the group $\mathcal{G}_\varphi$:

Let $I_F \subset W_F$ be the inertia group of F. We define

$$J_\varphi := \mathrm{Z}_{G^\vee}(\varphi(I_F)).$$

A root system:

Let $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{G^\vee} \in \mathfrak{B}^\vee(G)$. Recall $\mathcal{T} = \mathrm{Z}^\circ_{\mathcal{L}}$. Define $R(J^\circ, \mathcal{T})$ to be the set of $\alpha \in X^*(\mathcal{T})\setminus\{0\}$ which appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of J°_φ. It can be shown that $R(J^\circ, \mathcal{T})$ is a root system. We denote by $W^\circ_{\mathfrak{s}^\vee}$ its Weyl group.

An extended finite Weyl group:

Let $W_{\mathfrak{s}^\vee} := \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee)/L^\vee$. We have $W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \Gamma_{\mathfrak{s}^\vee}$, where

$$\Gamma_{\mathfrak{s}^\vee} := \left\{ w \in W_{\mathfrak{s}^\vee} \,:\, w(R(J^\circ, \mathcal{T})^+) \subset R(J^\circ, \mathcal{T})^+ \right\}.$$

A root datum:

We define a root datum

$$\mathcal{R}_{\mathfrak{s}^\vee} := (R_{\mathfrak{s}^\vee}, X^*(T_{\mathfrak{s}^\vee}), R_{\mathfrak{s}^\vee}^\vee, X_*(T_{\mathfrak{s}^\vee})),$$

where $T_{\mathfrak{s}^\vee} \simeq \mathfrak{s}_L^\vee = [L^\vee, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{L^\vee}$ and

$$R_{\mathfrak{s}^\vee} = \{m_\alpha \, \alpha \, : \, \alpha \in R(J^\circ, \mathcal{T})_{\mathrm{red}} \subset X^*(T_{\mathfrak{s}^\vee})\},$$

with $m_\alpha \in \mathbb{Z}_{>0}$. The group $W_{\mathfrak{s}^\vee}$ acts on $\mathcal{R}_{\mathfrak{s}^\vee}$.

Weight functions:

We define $W_{\mathfrak{s}^\vee}$-invariant functions

$$\lambda \colon R_{\mathfrak{s}^\vee} \to \mathbb{Q}_{>0} \quad \text{and} \quad \lambda^* \colon \{m_\alpha\alpha \in R_{\mathfrak{s}^\vee} : m_\alpha\alpha \in 2X_*(T_{\mathfrak{s}^\vee})\} \to \mathbb{Q}.$$

A twisted affine Hecke algebra:

The algebra $\mathcal{H} := \mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ is defined as

$$\mathcal{H}(G^\vee, \mathfrak{s}^\vee) := \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q^{1/2}) \rtimes \mathbb{C}[\Gamma_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}],$$

where $\natural_{\mathfrak{s}^\vee}$ is a certain 2-cocycle.

Theorem [A-Moussaoui-Solleveld, 2018]

There exists a canonical bijection

$$\begin{array}{ccc} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) & \longrightarrow & \mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)) \\ (\varphi, \rho) & \mapsto & M(\varphi, \rho) \end{array}$$

with the following properties

1. φ is bounded if and only if $M(\varphi, \rho)$ is tempered,
2. φ is discrete if and only if $M(\varphi, \rho)$ is an essentially discrete series and the rank of $R_{\mathfrak{s}^\vee}$ equals the dimension of $T_{\mathfrak{s}^\vee}/\mathfrak{X}_{\mathrm{nr}}({}^L G)$.

Theorem

If G is

1. an inner twist of $\mathrm{GL}_n(F)$ [A-Baum-Plymen-Solleveld, 2019]
2. a pure inner twist of quasi-split classical p-adic group [A-Moussaoui-Solleveld, 2022]
3. the group G_2 [A-Xu, 2022]

then, for every $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ such that $L \neq G$

$$\mathfrak{R}^{\mathfrak{s}}(G) \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{H}(G, \mathfrak{s})) \quad \text{with } \mathcal{H}(G, \mathfrak{s}) \cong \mathcal{H}(G^\vee, \mathfrak{s}^\vee)$$

where $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, \mathrm{LLC}(\sigma)]_{G^\vee}$.
In cases (1) (resp. (2)), the bijection

$$\mathcal{L}^G \colon \mathrm{Irr}^{\mathfrak{s}}(G) \xrightarrow{1-1} \mathrm{Irr}(\mathcal{H}(G, \mathfrak{s})) \xrightarrow{1-1} \mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)) \xrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$$

coincides with LLC defined by Harris-Taylor (resp. Arthur) for all $\mathfrak{s} \in \mathfrak{B}(G)$ (including the case $L = G$).

Remark

In all the cases (1), (2) et (3), the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Irr}^{\mathfrak{s}}(G) & \xrightarrow[1-1]{\mathcal{L}^G} & \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \\
\Big\downarrow {\scriptstyle \mathrm{Sc}} & & \Big\downarrow {\scriptstyle \mathrm{Sc}} \\
\mathrm{Irr}^{\mathfrak{s}_L}(L) & \xrightarrow[\mathrm{LLC}]{1-1} & \Phi_{\mathrm{e}}^{\mathfrak{s}_L^\vee}(L)
\end{array}$$

Conjecture [A.-Moussaoui-Solleveld, 2018]

Such a commutative diagram exists for every p-adic group G and all $\mathfrak{s} \in \mathfrak{B}(G)$.

Thank you very much for your attention!