The Wavefront Set & Arthur Packets for p-adic Groups

§1 Introduction

In the 80's Arthur introduced a series of conjs about automorphic forms. Let us recall the local implications.

k is local field, $G(k)$ the k-pts of conn. red. group split & defined over k. An Arthur param.

$$\psi : WD_k \times SL(2;\mathbb{C}) \longrightarrow {}^{\vee}G.$$

$\psi|_{WD_k}$ temp. $\psi|_{SL}$ is alg. From ψ can define

$$\phi_\psi : WD_k \longrightarrow {}^{\vee}G, \quad w \mapsto \psi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

The local content of Arthurs conj: for ψ we attach a $\Pi^A_\psi(G(k))$ consisting of irred adm. reps of $G(k)$

(1) $\Pi^L_{\phi_\psi} \subseteq \Pi^A_\psi$ (2) Π^A_ψ consists of unitary reps.

(3) stability, endoscopy . . .

For $k \in \{\mathbb{R}, \mathbb{C}\}$, the packets have been const. by ('92 Adams, Barbasch & Vogan) generalises work ('85 Barbasch, Vogan) where they construct "unip" Arthur packets for $\mathbb{C}$-case using WF-set.

P-adic has generally accepted defn π_ψ^A ('06, '14 Moeglin classical cases, '17 (CFMMX). Expected WF-set should play an important role.

Recent result of Waldsporger connects the WF to the LLC.

Thm: ('18 Waldspurger) Let $G=SO(2n+1,k)$, $k/\mathbb{Q}_p$ finite, (π,X) tempered, irred, w/ unipotent reduction. Let $\phi: W_k \times \underbrace{SL(2;\mathbb{C}) \longrightarrow {}^\vee G}$, be the L-param, let $d: \mathcal{N}_o^\vee \longrightarrow \mathcal{N}_o(\bar{k})$ the BLSV-dual, AZ denote Aubert-Zelevinsky dual. Then

$$WF(AZ(X)) = d(\mathbb{O}_X^\vee) \qquad \mathbb{O}_X^\vee = d\phi|_{SL}\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

Thm: ('21 Ciubotaru, Mason-Brown, O.) Let $G=\mathbf{G}(k)$, (π,X) irred in principal block, w/ "real inf char." Then

$$WF(AZ(X)) = d(\mathbb{O}_X^\vee).$$

Conj: Should true for ~~[illegible]~~ all irrep in principal except for some exceptions in E_7 & E_8.

Introduce a refinement of WF which is intimately related Arthurs conj.

§2 Preliminaries & Notation

Let $k/\mathbb{Q}_p$ finite, $\bar{k}/k$ alg. closure, K/k max^l unramified extn &

Let $\mathbb{G}$ ~~red~~ conn. red gp over $\mathbb{Z}$,

${}^\vee\mathbb{G}$ be Langlands dual gp $/\mathbb{Z}$.

$$\begin{tikzcd} k \arrow[r, hook] & K \arrow[r, hook] & \bar{k} \\ \mathfrak{o} \arrow[u, hook] \arrow[r, hook] \arrow[d] & \mathcal{O} \arrow[u, hook] \arrow[d] & \\ \mathfrak{f} \arrow[r] & \mathfrak{F}=\bar{\mathfrak{f}}. & \end{tikzcd}$$

$G = \mathbb{G}(\mathbb{C})$, ${}^\vee G = {}^\vee\mathbb{G}(\mathbb{C})\ldots$

Let $\mathcal{N}$ denote the functor $F \mapsto$ nilpotent elmts of $\mathfrak{g}(F)$. $\mathcal{N}_o$ be the functor $F \mapsto \mathcal{N}(F)/\mathbb{G}(F)$. Write $\mathcal{N}_o^{\mathbb{G}}$ to emphasize gp, $\mathcal{N}_o^\vee := \mathcal{N}_o^{{}^\vee\mathbb{G}}$, when $F=\mathbb{C}$ omit F. $\mathcal{N}_o$ for $\mathcal{N}_o(\mathbb{C})$. Let $d: \mathcal{N}_o \to \mathcal{N}_o^\vee$. When $F=\bar{F}$, view $\mathcal{N}_o(F)$ as a partially ordered set, $\mathbb{O}_1 \leq \mathbb{O}_2$, if $\mathbb{O}_1 \subseteq \overline{\mathbb{O}_2}$. When $F \in \{k, K\}$ use induced top.

<u>Ex</u>: $\mathbb{G} = GL(n)$, ${}^\vee\mathbb{G} = GL(n)$, $\mathcal{N}_o(\mathbb{C}) \leftrightarrow \{\lambda \vdash n\}$ w/ dominance order.

$d: \lambda \mapsto \lambda^t$.

<u>Lemma</u>: ('77 Pommerening) If $F=\bar{F}$, char$F$ is good for $\mathbb{G}$ then there is a canonical isom

$$\Theta_F : \mathcal{N}_o(F) \longrightarrow \mathcal{N}_o.$$

<u>Defn</u>: Let (π, X) be an admissible of $\mathbb{G}(k)$. By Harish-Chandra & Howe local char expn

$$\Theta_X = \sum_{\mathbb{O}\in\mathcal{N}_o(k)} c_{\mathbb{O}}(\pi)\, \hat{\mu}_{\mathbb{O}} \circ \exp^* \text{ on nbhd } V \ni 1$$

The (p-adic) WF-set of (π, X) is

$${}^k WF(X) = \max_{\mathbb{O}: c_{\mathbb{O}}(\pi)\neq 0} \mathbb{O}$$

The geometric WF-set

WF-set: $WF(x) = \max_{\mathcal{O}: c_{\mathcal{O}}(\pi)\neq 0} \mathcal{N}_o(\bar{k}/k)(\mathcal{O})$.

Conj: When (π, X) irred, ${}^{\bar{k}}WF(x)$ a singleton $\{\mathcal{O}\}$, & $\forall \mathcal{O}' \in {}^{k}WF(x)$, $\mathcal{N}_o(\bar{k}/k)(\mathcal{O}') = \mathcal{O}$.

Rmk: true for $\mathbb{R}$ & $\mathbb{F}_q$ case.

§3 The Canonical Unramified WF-set

We introduce refinement ${}^{K}WF$. It is designed to capture interplay b/w $\mathbb{Q}_p$ & $\mathbb{F}_q$ notions of WF.

↑ Kawanaka WF. ('92, Lusztig)

Let $\mathcal{B} = \mathcal{B}(\mathbf{G}(k))$ be the Bruhat-Tits building for $\mathbf{G}(k)$. Then for every face $c \subseteq \mathcal{B}$, there is a ses

$$\begin{tikzcd}
 & & & \downarrow & \\
1 \arrow[r] & U_c \arrow[r] & P_c \arrow[r] & M_c \arrow[r] & 1. \\
 & & \uparrow & \uparrow \ \mathfrak{f}\text{-pts of a comm. red. gp over } \mathfrak{f}.\ \mathbb{M}_c. &
\end{tikzcd}$$

Given $\mathbb{O} \in \mathcal{N}_o^{\mathbb{M}_c}(\mathfrak{f})$ the set

$$\{\mathbb{O} \in \mathcal{N}_o(k) : \mathbb{O} \cap (\mathbb{O} + \mathfrak{u}_c) \neq \emptyset\}$$

is non-empty & has a minimal elmt which we call $\mathcal{L}_c(\mathbb{O})$. ('02 DeBacker).

k	K /
$\mathfrak{o}$	$\mathcal{O}$
$\to \mathfrak{f}$	$\underline{\bar{\mathfrak{f}}}$

Analogously obtain a map

$$\mathcal{L}_c : \mathcal{N}_o^{\mathbb{M}_c}(\bar{\mathfrak{f}}) \longrightarrow \mathcal{N}_o(K).$$

Defn: Let (π, X) be an admissible rep of $\mathbf{G}(k)$, the unramified WF is f.d. Mc.

$$\overset{\text{thm}}{\longleftrightarrow} = \max_{\mathbb{O}: c_{\mathbb{O}}(\pi)\neq 0} \mathcal{N}_o(K/k)(\mathbb{O}).$$

$$^K\widetilde{WF}(X) = \max_{c\subseteq B} \mathcal{L}_c(^{\bar{\mathbb{F}}}WF(X^{U_c})) \subseteq \underbrace{\mathcal{N}_o(k)}.$$

Thm: ('21, O.) Let (π, X) be a depth 0 rep. Then

(1) $^{\bar{k}}WF(X) = \max \mathcal{N}_o(\bar{k}/k)(^K\widetilde{WF}(X))$.

(2) $^K\widetilde{WF}(X) = \max_{c\subseteq C_0} \mathcal{L}_c(WF(X^{U_c}))$.

Proof uses lifted generalized Gelfand-Graev reps from ('97 Barbasch & Moy)

§3.1 Unramified nilpotent orbits

Fix a $\mathbb{T}$ defined over $\mathbb{Z}$, let $\mathcal{A}$ be apte $\mathbb{T}(k)$.

Recall: A pseudo-levi is a conn. centralizer of a semisimple element.

Lemma: ('21, O.). There W-equiv. map

己: $\{\text{faces of } c\subseteq \mathcal{A}\} \longrightarrow \{(L, tZ_L^\circ): T\subseteq \underset{\subseteq G}{L}, C_G^\circ(tZ_L^\circ)=L$

己 has the following properties

(1) If 己$(c)=(L, tZ_L^\circ)$ the L is a ~~a~~ cx red gp w/ the same root data as M_c.

(2) 己$(c_1)=$ 己(c_2) iff $\mathcal{A}(c_1)+X^\vee = \mathcal{A}(c_2)+X^\vee$.

Main idea: $T_0 = \mathcal{A}/X^\vee$ $(\cong \mathbb{R}^n/\mathbb{Z}^n)$. There is a natural identification b/w $\hat{T_0} \simeq X(T)$. □

Let $I_K = \{(c, \mathbb{O}): c\subseteq B, \mathbb{O}\in \mathcal{N}_0^{M_c}(\bar{\mathbb{F}})\}$,

$\mathcal{N}_{o,c} = \{(\mathbb{O}, C): \mathbb{O}\in\mathcal{N}_o, C \overset{\text{cls}}{\subseteq} A(\mathbb{O})\}$. Then obtain

$$I_K \longrightarrow \mathcal{N}_{o,c}$$

$$(c,\mathbb{O}) \mapsto (L, \underline{tz_i^\circ}, \underset{\overset{\|}{\mathbb{O}}}{\Theta_f(\mathbb{O})}) \longmapsto (\mathrm{sat}_L^G \mathbb{O}, C).$$

$$\begin{tikzcd} (c,\mathbb{O}) \arrow[d, mapsto] & I_k \arrow[r] \arrow[d, two heads] & \mathcal{N}_{o,c} \\ \mathcal{L}_c(\mathbb{O}) & \mathcal{N}_o(k) \arrow[ur, dashed, "\sim", "\Theta"'] & \end{tikzcd}$$

Thm: ('21, O.) The map $\Theta: \underline{\mathcal{N}_o(k)} \longrightarrow \underline{\mathcal{N}_{o,c}}$ is a bij &

$\mathbb{O}_1 \leq \mathbb{O}_2$ $\leq$

$$\begin{tikzcd} \mathcal{N}_o(k) \arrow[r, "\Theta"] \arrow[d] & \mathcal{N}_{o,c} \arrow[d, "pr_1"] \\ \mathcal{N}_o(\bar{k}) \arrow[r, "\Theta_{\bar{k}}"] & \mathcal{N}_o \end{tikzcd}$$

$\leq_A$ $\mathbb{O}_1 \leq_A \mathbb{O}_2$?

Recall: $d_S: \mathcal{N}_{o,c} \longrightarrow \mathcal{N}_o^\vee$ so using Θ, obtain a map $d_S \mathcal{N}_o(k) \to \mathcal{N}_o^\vee$. ('03, Achar) used d_S to define a pre-order on $\mathcal{N}_{o,c}$, $(\mathbb{O}_1, C_1) \leq_A (\mathbb{O}_2, C_2)$ if $\mathbb{O}_1 \leq \mathbb{O}_2$ and $d_S(\mathbb{O}_1, C_1) \geq d_S(\mathbb{O}_2, C_2)$. Θ is <u>not</u> an isom of pre-orders. The equiv. classes $\sim_A$

$$\mathcal{Q}: \mathcal{N}_{o,c} \longrightarrow \mathcal{N}_{o,\bar{c}} = \{(\mathbb{O}, \bar{C}) : \mathbb{O} \in \mathcal{N}_o, \bar{C} \overset{cds}{\leq} \bar{A}(\mathbb{O})\}$$

are fibres of this. Use Θ to transfer $\sim_A$

$$(\mathcal{N}_o(k)/\sim_A, \leq_A) \simeq (\mathcal{N}_{o,\bar{c}}, \leq_A).$$

Defn: Let (π, X) be an adm repn of $\mathbf{G}(k)$.

Then $\quad {}^K WF(X) := \max_{c \in B} [\mathcal{L}_c(WF(X^{U_c}))] \subseteq \mathcal{N}_o(k)/\sim_A.$

Conj: (1) ${}^K WF(x)$ is a singleton.

If this is true then ${}^K WF(x) = ({}^{\bar{k}} WF(x), \bar{C})$

§ 3.2 Computing KWF

Let $R_I(G(k))$ (rep. $R(\widetilde{W})$, $\widetilde{W} = W \ltimes X^\vee$) denote the Grothendieck gp of principal block (resp. f.d. $\mathbb{C}$-reps). There

$$(-)_{q\to 1} : R_I(G(k)) \longrightarrow R(\widetilde{W}).$$

For $c \in \mathcal{B}$, let $W_c \leq \widetilde{W}$ generated by reflns in hyperplanes through c.

For $E \in \mathrm{Irr}(W_c)$, let $\mathbb{O}(E)$ denote $\in \mathcal{N}_o^{M_c}(\bar{\mathbb{F}})$ the Springer supp of $\tilde{E}$ the unique special in family of $E \otimes \mathrm{sgn}$.

Thm: ('21, O., '21 CMO) $\mathcal{L}_c(WF(X^{U_c})) = \{\mathcal{L}_c(\mathbb{O}(E)) : E \subseteq X_{q\to 1}\downarrow_{W_c}\}$.

$$\widetilde{W} \dashrightarrow W, \quad W_c \longrightarrow \dot{W}_c$$

Defn: $E \in \mathrm{Irr}(W)$, & $WF_c(E) = \{ \mathcal{Q} \circ \Theta \circ \mathcal{L}_c(\mathbb{O}(E')) : E' \subseteq E\downarrow_{\dot{W}_c} \} \subseteq \mathcal{N}_{o,\bar{c}}$. Define $WF(E) = \max\limits_{c \subseteq c_o} WF_c(E)$.

$c \subseteq c_o \leftrightarrow J \subsetneq \widetilde{\Delta}$.

Lemma: ('21 CMO) Let $E \in \mathrm{Irr}(W)$. Then

$$WF(E) = \underset{\uparrow}{d}_A(\mathbb{O}_E^\vee, 1).$$

← springer supp of E w.r.t. ${}^\vee G$.

Achar duality map $\mathcal{N}_{o,\bar{c}}^\vee \to \mathcal{N}_{o,\bar{c}}$

Thm: ('21 CMO) Let (π, X) be Iwahori-spherical w/ L-param $\phi : \overbrace{W_k \times SL(2;\mathbb{C})}^{WD_k} \longrightarrow {}^\vee G$, & $\underbrace{\phi(Fr, \begin{pmatrix} q^{1/2} & \\ & q^{-1/2} \end{pmatrix})}_{} \in {}^\vee T_{\mathbb{R}>0}$. Then ${}^{\bar{K}}WF(AZ(X))$ is a singleton

s. $\Big({}^{K}WF(AZ(X)) = d_A(\mathbb{O}^\vee_X, 1).$ $\leftarrow$

$\underbrace{{}^{\bar{\mathbb{R}}}WF(AZ(X))}_{} = \underline{d(\mathbb{O}^\vee_X)}. \Big)$ $(\underset{\tau}{\mathbb{O}'}, \overset{d}{\bar{C}'}).$

$\mathcal{N}_o \simeq \mathcal{N}_o(\bar{\mathbb{R}})$

$$E_{\mathbb{O}^\vee,1} = j^{W}_{W_c} E_{\mathbb{O},1} \qquad \mathbb{O} \text{ special}$$

$$\psi: \underbrace{\overset{\text{triv.}}{W_\mathbb{R}} \times SL(2;\mathbb{C})} \overset{\curvearrowright}{\times} SL(2;\mathbb{C}) \longrightarrow {}^\vee G$$

$$AZ(\Pi^A_\psi) = \Pi^A_{\psi^t}$$

$$\Pi^A_\psi = AZ$$

$\Pi^A_{\mathbb{O}^\vee} = \{ X : X$ has inf char $\frac{1}{2}h^\vee$, ${}^{K}WF(X) = d_A(\mathbb{O}^\vee, 1)\}$
consists of anti-tempered reps.

$\Pi^{A,weak}_{\mathbb{O}^\vee} = \{ X$ ..... - ----. $WF(X) = d(\mathbb{O}^\vee)\}.$

union of A-packets?

$\supseteq \Pi^A_{\mathbb{O}^\vee}.$