On the formal degree conjecture for classical groups

§1 - $\underline{H}$ $/_{F\,:\,\text{local field}}$ conn. reductive + pure inner form of a quasi-split gp (for simplicity)

$H = \underline{H}(F)$, $\psi : F \longrightarrow \mathbb{C}^\times$ non trivial

σ : disc. series of H $\longrightarrow$ $d_H(\sigma) \in \mathbb{R}_{>0}$ formal degree

$u, v \in \sigma$
$u^\vee, v^\vee \in \sigma^\vee$

$$\int_{H/A_H} \langle \sigma(h) u, u^\vee \rangle \langle v, \sigma^\vee(h) v^\vee \rangle \, \underbrace{dh}_{|\omega_H|_\psi} = \frac{\langle u, v^\vee \rangle \langle v, u^\vee \rangle}{d_H(\sigma)}$$

$\iota : \underline{H}_{\bar F} \xrightarrow{\sim} \underline{H}_{\mathbb{Z}} \times \bar F$

$\omega_H = \iota^* \omega_{\mathbb{Z}}$

LLC : $\sigma \in \mathrm{Irr}(H) \longmapsto \phi_\sigma : \mathcal{L}_F \longrightarrow {}^L H = \hat H \rtimes \Gamma$

+ ρ_σ irrep of $S_\sigma = \pi_0(\mathrm{Cent}_{\hat H}(\phi_\sigma))$

$\mathrm{Ad}_H : {}^L H \curvearrowright \mathrm{Lie}(\hat H) / \mathrm{Lie}(Z(\hat H)^\Gamma)$

$$\gamma(s, \sigma, \mathrm{Ad}_H) = \varepsilon(s, \mathrm{Ad}_H \circ \phi_\sigma, \psi) \frac{L(1-s, \mathrm{Ad}_H \circ \phi_\sigma)}{L(s, \mathrm{Ad}_H \circ \phi_\sigma)}$$

Conj 1 (Hiraga - Ichino - Ikeda):

$$d_H(\sigma) = \frac{\dim(\rho_\sigma)}{|S_\sigma|} |\gamma(0, \sigma, \mathrm{Ad}_H)|$$

for every disc series.

Rmks: • $\dim(\rho_\sigma)$ shouldn't depend on anything.

- For classical groups, $\dim(\rho_\sigma) = 1$.

Plancherel formula:

"elementary" measure (coming from Haar measures on tori of unr chars)

$$f \in C_c^\infty(H), \quad f(1) = \int_{\mathrm{Temp}(H)} \Theta_\sigma(f) \, \mu_H(\sigma) \, d\sigma$$

$\mathrm{Temp}(H) = \{\text{tempered irreps of } H\}/_{\text{iso}}$, $\Theta_\sigma(f) = \mathrm{Tr}\,\sigma(f)$, $\mu_H(\sigma)$ Plancherel density

For σ d.s., $\mu_H(\sigma) = d_H(\sigma)$

In g^{al}, $\mu_H(I_L^G(\tau)) = d_L(\tau) \times$ product of intertwining operators (Harish-Chandra)

I_L^G: parabolic induction

Conj 1 + Langlands conj on normalization of IO $\Rightarrow$

Conj 2:

$$\mu_H(\sigma) = \frac{\dim(\rho_\sigma)}{|S_\sigma|} \, |\gamma^*(0, \sigma, Ad_H)|$$

γ^*: leading term in Taylor expansion

for a.a. $\sigma \in \mathrm{Temp}(H)$.

(char $F = 0$)

Conj 1/2 known for
- $F = \mathbb{R}$ (Harish-Chandra)
- GL_n (Silberger, Zink + Shahidi)
- σ d.s. stable of $U(2n)$ (H II)
- $SO(2n+1)$ (Ichino-Lapid-Mao)
- $U(n)$ (B.-P., Morimoto)
- regular / non-singular supercusp. (D. Schwein / K. Ohara)
- unipotent d.s. (Feng - Opdam - Solleveld).

Thm 1: Conj 1&2 hold for $H = Sp(2n)$ or $SO(2n)$

Rmks: • The LLC for $SO(2n)$ is only known up to $O(2n)$-conj. (Arthur, Moeglin) but Conj 1/2 are insensible to this

- For those groups Conj 1 $\Rightarrow$ Conj 2 by Arthur
- Proof similar to H11 argument for stable d.s. of $U(2n)$

combine
- Twisted endoscopic char. of L-packets
- Shahidi's idea of relating residues of IO to twisted orbital integrals

§2. Twisted endoscopy

$\underline{H} = Sp(2n)$ or $SO(2n)$ $\underline{\text{split}}$

V: N-dim'l v.s. $/F$, $\quad M = GL(V)$, $A = Z(M)$

$$\tilde{M} = \mathrm{Isom}_F(V, V^*) \simeq \left\{\begin{array}{l}\text{non-deg} \\ \text{bil. forms on } V\end{array}\right\} \quad \underline{\text{twisted}} \text{ group}$$

(left & right M-torsor)

H is a twisted endoscopic group of $\tilde{M}$ for $N = \begin{cases} 2n+1, & H = Sp(2n) \\ 2n, & H = SO(2n) \end{cases}$

- A correspondence: $\quad H_{rs}/_{\text{stab conj}} \longleftrightarrow \tilde{M}_{rs}/_{M\text{-conj}}$ $\quad$ rs: regular semisimple

  $\delta \longleftrightarrow \gamma$

- A transfer: $\quad C_c^\infty(H) \ni f^H \longleftrightarrow f^{\tilde{M}} \in C_c^\infty(A \backslash \tilde{M})$

st

$$\underbrace{SO(\delta, f^H)}_{\text{stable orb. int.}} = \sum_{\delta \leftrightarrow \gamma} \Delta(\delta, \gamma) \underbrace{O(\gamma, f^{\tilde{M}})}_{\text{twisted orb. integ.}}$$

<u>Functorial</u> lift: $\quad \mathrm{Irr}(H) \longrightarrow \mathrm{Irr}(\tilde{M}) = \left\{ (\tilde{\pi}, \pi) \;\middle|\; \begin{array}{l} \pi : M \to GL(E_\pi) \text{ irrep}, \\ \tilde{\pi} : \tilde{M} \to GL(E_\pi) \\ \tilde{\pi}(m_1 \gamma m_2) = \pi(m_1)\tilde{\pi}(\gamma)\pi(m_2) \end{array} \right\}$

$\sigma \longmapsto \tilde{\pi}$

where the L-param of π:

$$\phi_\pi : L_F \xrightarrow{\phi_\sigma} \hat{H} = SO_N(\mathbb{C}) \subset GL_N(\mathbb{C})$$

and the ext. $\tilde{\pi}$ is Whittaker normalized

Arthur / Moeglin: $\quad$ For $\quad f^H \longleftrightarrow f^{\tilde{M}}$

$$\Theta_{\tilde\pi}(f^{\tilde M}) = \frac{|S_\sigma^+|}{2|S_\sigma|}\sum_{\sigma \to \tilde\pi} \Theta_\sigma(f^H) \quad \Big]\ \text{character relations.}$$

$$S_\sigma^+ = \pi_0\left(\mathrm{Cent}_{O_N(\mathbb{C})}(\phi_\sigma)\right)$$

§3 - The proof

Let $\gamma \in \tilde M$ be a nondeg. sum of
- symplectic form of max^l rk
- quad form of rk 1

e.g. $\gamma = \begin{pmatrix} & & 0 \\ & \cdot^{\cdot^{\cdot}} & \\ -1 & 0 & \\ 1 & & 1 \end{pmatrix}$ N even

$\gamma = \left(\begin{array}{c|c} \begin{matrix} & 0 \\ -1 & 0 \\ 1 & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right)$ N odd

<u>Rmks</u>:
- γ is unique up to M-conj. modulo the center
- γ is semisimple if N odd but not if N even.

<u>Thm 2</u>: For $f^{\tilde M} \in C_c^\infty(\tilde M / A)$

$$O(\gamma, f^{\tilde M}) = \int_{\mathrm{Temp}(H)/\mathrm{Stab}} \Theta_{\tilde\pi}(f^{\tilde M}) \frac{2}{|S_\sigma^+|} \gamma^*(0,\sigma,\mathrm{Ad}\,\phi_H)\, d\sigma$$

$\sigma \to \tilde\pi$

$\| \qquad \int_{M_\gamma \backslash M} f^{\tilde M}(m^{-1}\gamma m)\frac{dm}{dm_\gamma}$

$\mathrm{Temp}(H)/\mathrm{Stab} \uparrow\uparrow \mathrm{Temp}(H)/\sim$

$\sigma_1 \sim \sigma_2 \iff \sigma_1, \sigma_2$ have the same lift to M

<u>Thm 2 $\Rightarrow$ Thm 1</u>
(Shelstad/S. Varma) For $f^H \leftrightarrow f^{\tilde M}$

$$O(\gamma, f^{\tilde M}) = f^H(\varepsilon) \qquad \varepsilon = (-1)^{N+1} \in Z(H)$$

// Thm2 $\qquad$ \\ Planch.

$$\int_{\mathrm{Temp}(H)/\mathrm{Stab}} \dots \qquad \int_{\mathrm{Temp}(H)} \Theta_\sigma(f^H)\, \omega_\sigma(\varepsilon)\mu_H(\sigma)\, d\sigma$$

+ unicity of Plancherel measure

On the proof of thm 2 : $W = V \oplus V^* \oplus F$

$$q(v, v^\vee, \lambda) = \langle v, v^\vee \rangle + \lambda^2$$

Set $G = SO(W, q) \supset P = \mathrm{Stab}_G(V) = MU$

$$\overline{P} = \mathrm{Stab}_G(V^*) = M\overline{U} = wPw^{-1}$$

$$M = P \cap \overline{P} \simeq GL(V)$$

Fix $w \in \mathrm{Norm}_G(M) \setminus M$ then $\tilde{M} = Mw$

$w^2 = 1$

→ For $\pi \in \mathrm{Irr}(M)$, $M(\pi, s) : I_P^G(\underbrace{\pi \otimes |\det|^{s/2}}_{\pi_s}) \longrightarrow I_{\overline{P}}^G(\pi_s)$

std intertwining op. $e \longmapsto \int_{\overline{U}} e(\bar{u}\,\cdot)\, d\bar{u}$

→ For π supercusp., Shahidi ('92) computes the residue of $M(s, \pi)$ at $s = 0$ as $O(\gamma, f_{\tilde{\pi}})$ matrix coeff of some ext. $\tilde{\pi}$ of π

→ Idea : apply the same computation to

$$\Pi = \text{regular rep}^{n} \text{ on } C_c^\infty(M/A)$$

More precisely

$$s \longmapsto f_s \in I_P(\Pi_s) \overset{\sim}{=} \int^{\oplus}_{\mathrm{Temp}(M)} I_P(\pi_s \otimes \check{\pi}_s) \;\cong\; C_c^\infty\left(\frac{G}{AU}, \delta_P^{1/2} |\det|^{s/2}\right)$$

a "nice" holomorphic section.

$$O(\gamma, f^{\tilde{M}}) \overset{?}{=} \operatorname*{Res}_{s=0} \int_{\overline{U}} f_s(\bar{u})\, d\bar{u} = \int_{\mathrm{Temp}(H)/\mathrm{Stab}} h_{\tilde{\pi}}(f^{\tilde{M}}) \frac{2}{|S_\sigma^+|} \gamma^*(0, \sigma, \mathrm{Ad}_H)\, d\sigma$$

where $f^{\tilde{M}} = \delta_P^{-1/2} f_0 |_{\tilde{M}}$