Wave-front set and graded Springer theory

arxiv: 2207.13445

§1 Wave-front set and degenerate Whittaker functional

§2 More history and results

§3 Graded Lie algebra

G connected reductive gp. $/F$ non-arch. local field with res. field k. Fix $\ell \neq \mathrm{char}(k)$ and all rep'ns / functions are over $\overline{\mathbb{Q}}_\ell$. "Pretend" $\mathrm{char}(k) \gg \mathrm{rk}\, G$, $\mathrm{char}(F) = 0$.

$\overline{\mathbb{Q}}_\ell \overset{\sim}{=} \mathbb{C}$

Fix $\psi : F \to \overline{\mathbb{Q}}_\ell^\times$ non-trivial char.

Write $\mathcal{O}(0)_{\mathfrak{g}^*(F)}$:= set of nilpotent orbits in $\mathfrak{g}^*(F)$

$$\pi \in \mathrm{Irr}^{\mathrm{adm}}(G(F)) \underset{\mathrm{HC}}{\xrightarrow{\text{Howe}}} \Theta_\pi(\exp(X)) = \sum_{\mathcal{O} \in \mathcal{O}(0)_{\mathfrak{g}^*(F)}} c_{\mathcal{O}}(\pi)\, \hat{I}_{\mathcal{O}}(X)$$

$c_{\mathcal{O}}(\pi) \in \overline{\mathbb{Q}}_\ell$

$$\hat{I}_{\mathcal{O}}(f) = \int_{\mathcal{O}} \hat{f}$$

Def 1 $\quad WF^{\mathrm{geom}}(\pi) := \bigcup_{c_{\mathcal{O}}(\pi) \neq 0} \overline{\mathcal{O}}^{\mathrm{Zar}} \underset{\text{Zar closed}}{\subset} \mathfrak{g}^*$

Thm (Moeglin-Waldspurger '87) Suppose $\mathcal{O} \underset{\text{open}}{\subset} WF^{geom}(\pi)$

Then $\forall \lambda: \mathbb{G}_m \to G$, $e \in \mathfrak{g}^*(F)$ with $\underbrace{Ad(\lambda(t))e = t^2 e}$

$\Updownarrow$

$e \in {}_2^{\lambda}\mathfrak{g}^*(F)$

$\exists$ non-zero $\phi: \pi \to \overline{\mathbb{Q}}_\ell$ s.t. $\phi(\exp(X).v)$

(degenerate Whittaker functional) $= \psi(\langle e, X\rangle)\phi(v)$

$\forall X \in \underbrace{{}_{\leq -2}^{\lambda}\mathfrak{g}(F)}_{\text{actually larger}}$, $v \in \pi$

When $\mathcal{O}$ is regular, this was by Rodier ('75)

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§2 More history and results

(a) [MW '87] For $G = GL_n$, $WF^{geom}(\pi)$ is irred and can be described in terms of the Bernstein-Zelevinsky data.

(b) [Kawamata '87, Lusztig '92, Achar-Aubert '07, Taylor '16] <u>Fact</u> For $\rho \in Irr(G(k))$ $(k \cong \mathbb{F}_q)$

$$\Theta_\rho(\exp(X)) = \sum_{\mathcal{O}\in\mathcal{O}(0)_{\mathfrak{g}^*(k)}} c_{\mathcal{O}}(\rho)\,\hat{I}_{\mathcal{O}}(X), \ \forall X \in \mathfrak{g}^{nil}(k)$$

$(\exists\, c_{\mathcal{O}}(\rho) \in \overline{\mathbb{Q}}_\ell)$

<u>Thm</u> $\exists\, \mathcal{O}_\rho \in \mathcal{O}(0)$ s.t. $c_{\mathcal{O}_\rho}(\rho) \neq 0$ and

for $\mathcal{O} \in \mathcal{O}(0)$ s.t. $c_{\mathcal{O}}(\rho) \neq 0 \Rightarrow \mathcal{O} \subset \overline{\mathcal{O}_\rho}^{Zar}$.

$\mathcal{O}_\rho$ can be computed in terms of the Lusztig's data ρ.

Many works by Barbasch, Gomez, Gourevitch, D. Jiang, B. Liu, D. Liu, Moeglin, Moy, Sahi, Savin, L. Zhang, ...

(c) (Waldspurger '18, '20) $G = SO_{2n+1}$, $\pi \in Irr^{adm}(G(F))$ unipotent tempered / anti-tempered

$\Rightarrow WF^{geom}(\pi)$ is irred and is computable in terms of Lusztig's data

(d) [Ciubotaru–Mason-Brown–Okada '21, '22, Aizenbud–Gourevitch–Sayag '22) $\pi \in \mathrm{Irr}^{adm}(G(F))$ depth-0 supercuspidal or Iwahori-spherical.

$\Rightarrow WF^{geom}(\pi)$ is irred and ——.

<u>Conj.</u> For $\pi \in \mathrm{Irr}^{adm}(G(F))$, $WF^{geom}(\pi)$ is irreducible.

Very analogous result is true when $F = \mathbb{R}$.

(Barbasch–Vogan '80, Joseph)

(e) <u>Thm 1</u> (T. '22) $G = U_7$ ramified unitary gp

$\exists\, \pi \in \mathrm{Irr}^{adm}(\pi)$, $\mathrm{dep}(\pi) = n + \frac{1}{2}$ s.t.

$$WF^{geom}(\pi) = \overline{\mathcal{O}}^{2ar}_{511} \cup \overline{\mathcal{O}}^{2ar}_{43}$$

$k = \mathbb{F}_3$

$k = \mathbb{F}_5$

Guess: For Kaletha's reg. supercusp.

if $|k| \gg e^{\mathrm{rank}\, G}$, then $WF(\pi) = \overline{\mathcal{O}}_{reg}$ or $\overline{\mathcal{O}}_{subreg}$

very wrong; depth-0 supercuspidal for Sp_{2n} can have $WF^{geom}(\pi) = \overline{\mathcal{O}}_{2n_1, 2n_2}$, any $n_1 + n_2 = n$.

(e') $H = GL_{7/k}$, $\theta : H \to H$, $h \mapsto (h^t)^{-1}$, $(H^\theta)^\circ = SO_7 =: G$

$V := (\mathrm{Lie}\, H)^{\theta = -1} \cong \mathrm{Sym}^2(k^7)$, $G \curvearrowright V$ s.t. $V \cong V^*$.

For $A \in V^*(k)$, $X \in V(k)$, write

$$\hat{I}_A(X) = \sum_{g \in G(k)} \psi(\langle A, g.X \rangle)$$

<u>Fact</u> $\exists \{c_{\mathcal{O}}(A)\}_{\mathcal{O} \in \mathcal{O}(0)_{V^*(k)}}$ s.t.

$$\hat{I}_A(X) = \sum_{\mathcal{O} \in \mathcal{O}(0)_{V^*(k)}} c_{\mathcal{O}}(A) \cdot \hat{I}_{\mathcal{O}}(X) \quad \forall X \in V^{nil}(k)$$

Write $WF^{geom}(A) = \bigcup_{c_{\mathcal{O}}(A) \neq 0} \overline{\mathcal{O}}^{2ar}$

<u>Prop 2</u> $\exists A \in V^*(k)$ $(k = \mathbb{F}_3$ or $\mathbb{F}_5)$ s.t. $WF^{geom}(A) = \overline{\mathcal{O}}_{511}^{2ar} \cup \overline{\mathcal{O}}_{43}^{2ar}$

<u>Thm 3</u> Modulo technical issues

(known to hold in arxiv: 2207.13445)

Prop 2 $\Rightarrow$ Thm 1.

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§3 Graded Lie algebra $\qquad \mathrm{Lie}\, H = (\mathrm{Lie}\, G) \oplus V$

$\underline{\text{Def}^{\text{n}}}$ $(V^*)^{rs} := \{ v \in V^* \mid G.v$ closed and $\mathrm{Stab}_G(v)$ finite $\}$

$\underline{\text{Rmk}}$ (1) $(V^*)^{rs} \subset V^*$ is open dense

(2) V^{nil} contains reg. nilpotent.

(3) $\theta : H \to H$ is the "Cartan invol. for split $GL_7(\mathbb{R})$"

Let $C(V^{nil}) := \{ f \in V^{nil}(k) \to \overline{\mathbb{Q}}_\ell \mid f$ is $G(F)$-inv.$\}$

Say $f \in C(V^{nil})$ has small co-support if

$\mathrm{supp}(\hat{f}) \cap V^{rs}(k) = \emptyset$.

Write $C(V^{nil})^{sm\text{-}cs} \subset C(V^{nil})$ those f.

$\underline{\text{Lemma 4}}$ For $\mathcal{O} \in \mathcal{O}(0)_{V^*}$, $A \in V^*(k)$, TFAE

(i) $\mathcal{O} \not\subset WF^{geom}(A)$

(ii) $\{ f \in C(V^{nil}) \mid \overline{G.n}^{2ar} \supset \mathcal{O} \Rightarrow \hat{f}(n) = 0 \} = C(V^{nil})^{(\mathcal{O})}$

$\subset \{ f \in C(V^{nil}) \mid \hat{f}(A) = 0 \}$

Following lemma 4, want to find generators

for $(*)_{(\mathcal{O})}$ $\underline{C(V^{nil})^{(\mathcal{O})} / C(V^{nil})^{(\mathcal{O})} \cap C(V^{nil})^{sm-cs}}$

In our case of $H^{\mathfrak{S}\theta}$, $G = (H^{\theta})^{0}$, $V = (\mathrm{Lie}\, H)^{\theta = \tau}$, ...

$\exists$ proj. smooth families of var. (Hessenberg var.)

over $(V^{*})^{rs}$ $X^{0}, X^{1}, X^{2}, X^{2'}, X^{3}$ s.t.

$$\left\{\begin{array}{ll} \underline{X_A^0(k) = \phi} \Longleftrightarrow \text{Lemma 4 for } \mathcal{O}_7 \\ \underline{X_A^1(k) = \phi} \Longleftrightarrow \cancel{\text{lem}} \quad \text{for } \mathcal{O}_{61} \\ \underline{X_A^2(k) = \phi} \quad \text{for } \mathcal{O}_{52} \\ X_A^{2'}(k) = \phi \quad \text{for } \mathcal{O}_{511} \\ X_A^{3}(k) = \phi \quad \text{for } \mathcal{O}_{43} \end{array}\right.$$

$\Leftarrow$ can be $\underline{\text{systematically}}$ $\underline{\text{produced}}$

X

$\mu_2^3 \overset{\bar{k}}{\cong} \mathrm{Stab}_G(A) \curvearrowright X_{A'}^0, X_A^1, X_A^2$ free

$A \overset{\mathrm{stably}}{\sim} A'$ $\quad X_{A'}^{i} = X_A^{i} \times^{\mathrm{Stab}_G(A)} \mathrm{inv}(A, A')$