Kazhdan-Lusztig Equivalence at the Iwahori Level

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Overview

1. Statement of Result
2. Proof Strategy: Factorization
3. Quantum Side
4. Affine Side
5. Global Methods

Notations

G	Reductive group over $\mathbb{C}$ (for this talk, assumed simple)
$G(O), G(K)$	Arc (resp. loop) group of G
$\mathfrak{g}$	Lie algebra of G
$h^\vee$	Dual Coxeter number
$\check{\Lambda}, \Lambda$	Weight lattice / coweight lattice
W	Weyl group for G
κ	Non-degenerate W-invariant symmetric bilinear form on Λ
$\check{\kappa}$	Corresponding bilinear form on $\check{\Lambda}$
c	$\check{\kappa} = \dfrac{c - h^\vee}{2h^\vee} \check{\kappa}_{\text{min}}$, where $\check{\kappa}_{\text{min}}(\check{\alpha}_i, \check{\alpha}_i) = 2$ for long roots $\check{\alpha}_i$

Kazhdan-Lusztig Equivalence

Theorem ([KL94])

If $c \in \mathbb{C} \setminus \mathbb{Q}$, or $c \in \frac{m}{n} \in \mathbb{Q}^{<0}$ for $(m, n) = 1$ and m not too small, then there exists a braided monoidal equivalence $\mathrm{KL}_\kappa(G)^\heartsuit \simeq \mathrm{Rep}_q(G)^\heartsuit$.

- $\hat{\mathfrak{g}}_\kappa$ Central extension of $\mathfrak{g}((t))$ given by the 2-cocycle κ
- $\mathrm{KL}_\kappa(G)^\heartsuit$ Abelian category of finitely generated, smooth, $G(O)$-integrable $\hat{\mathfrak{g}}_\kappa$-modules at level κ
- $U_q^{\mathrm{Lus}}(\mathfrak{g})$ Lusztig's quantum group specialized at $q := e^{\frac{\pi i}{rc}}$, where r is the lacing number of $\mathfrak{g}$
- $\mathrm{Rep}_q(G)^\heartsuit$ Abelian category of finite dimensional $\check{\Lambda}$-graded $U_q^{\mathrm{Lus}}(\mathfrak{g})$-modules, where $K_{\check{\alpha}_i} \in U_q^{\mathrm{Lus}}(\mathfrak{g})$ acts via grading

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At generic levels, both are equivalent to $\mathfrak{g}\text{-mod}^B$. Rational levels are more interesting.

Main Result

Theorem (Lin Chen and C.F.; Conjectured by D. Gaitsgory)

If $c \in \mathbb{C} \setminus \mathbb{Q}$, or $c \in \frac{m}{n} \in \mathbb{Q}$ for $(m, n) = 1$ and m not too small, then there exists an equivalence of (DG) categories

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- The proof is independent from the original one by K-L. Comparison with K-L is ongoing work;
- The RHS carries a braided monoidal structure (compatible with $\text{Rep}_q(G)^\heartsuit$); consequently it equips LHS with a (previously unknown) braided monoidal structure. We do not yet know how to describe it explicitly.

Proof Strategy

The following strategy works (only) for $c > 0$. The $c < 0$ case follows formally via categorical duality.

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\begin{array}{ccc}
\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\text{ren}} & \dashrightarrow & \text{Rep}^{\text{mxd}}_q(G)_{\text{ren}} \\
{\scriptstyle J^{\text{KM}}_*}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle J^{\text{Quant}}_*} \\
\Omega^{\text{KM}}\text{-FactMod}_{\text{alg}} & \xrightarrow[\text{Riemann-Hilbert}]{\simeq} & \Omega^{\text{Quant}}\text{-FactMod}_{\text{top}}
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In general, given a lax monoidal functor $F : C \to D$ between monoidal categories, it automatically factors as

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Factorization Objects

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At the level of !-fibers, such an object gives, among other things, a vector space $\iota^!_{\check{\lambda}\cdot x}(A)$ for every $\check{\lambda} \in \check{\Lambda}^{<0}$, $x \in \mathbb{A}^1(\mathbb{C})$.

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$*$-fibers encode a *coalgebra* structure on $A_{\mathrm{coalg}} := \bigoplus_{\check{\lambda}\in\check{\Lambda}^{<0}} \iota^*_{\check{\lambda}\cdot x}(A)$.

Similarly, a $\check{\Lambda}$-graded *factorization module* M (supported at $0 \in \mathbb{A}^1$) over A is (among other data) a sheaf on the moduli space

$$\mathfrak{Conf}_0 := \{\check{\lambda}_0 \cdot 0 + \sum_{i \in I, |I| < \infty} \check{\lambda}_i \cdot x_i \mid \check{\lambda}_0 \in \check{\Lambda}, \check{\lambda}_i \in \check{\Lambda}^{<0}, x_i \text{ disjoint}, x_i \neq 0\};$$

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As above, this encodes an A_{alg}-module structure on $\bigoplus_{\check{\lambda}_0 \in \check{\Lambda}} \iota^!_{\check{\lambda}_0 \cdot 0}(M)$ and an A_{coalg}-comodule structure on $\bigoplus_{\check{\lambda}_0 \in \check{\Lambda}} \iota^*_{\check{\lambda}_0 \cdot 0}(M)$.

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Riemann-Hilbert allows the comparison between algebraic factorization modules (using D-modules) and topological ones (using constructible sheaves).

Mixed Quantum Groups

Recall that both the Lusztig algebra $U_q^{\mathrm{Lus}}(\mathfrak{n})$ and the Kac-De Concini algebra $U_q^{\mathrm{KD}}(\mathfrak{n})$ can be realized as Hopf algebras *internal* to $\mathrm{Rep}_q(T)^{\heartsuit}$.

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$$\Delta(E_{\check{\alpha}_i}) = E_{\check{\alpha}_i} \otimes 1 + 1 \otimes E_{\check{\alpha}_i} \quad E_{\check{\alpha}_i} \in U_q^{\mathrm{Lus}}(\mathfrak{n}).$$

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The abelian category $\mathrm{Rep}_q^{\mathrm{mxd}}(G)^{\heartsuit}$ consists of $V \in \mathrm{Rep}_q(T)^{\heartsuit}$ with a *locally nilpotent* $U_q^{\mathrm{Lus}}(\mathfrak{n})$ action and a compatible (arbitrary) $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$ action.

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$\mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{ren}}$ is the ind-completion of

$$\{V \in D^b(\mathrm{Rep}_q^{\mathrm{mxd}}(G)^\heartsuit) \text{ s.t.}$$

$$\mathrm{oblv}(V) \in U_q^{\mathrm{KD}}(\mathfrak{n}^-)\text{-mod}(D(\mathrm{Rep}_q(T)^\heartsuit)) \text{ is compact}\}.$$

Proposition

There exists a topological factorization algebra Ω^{Quant} *and an equivalence of DG categories* $J_*^{\mathrm{Quant}} : \mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{ren}} \simeq \Omega^{\mathrm{Quant}}\text{-}\mathrm{FactMod}_{\mathrm{top}}$.

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Due to the need of working derivedly, we go down an entirely different path, using (homotopical) Koszul duality and the language of $\mathbb{E}_2$-algebras, building upon the foundational work of Lurie in [Lur].

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Remark

$\iota^!_{\check{\lambda}\cdot 0}(J_*^{\mathrm{Quant}}(M))$ is the $\check{\lambda}$-component of $\mathrm{Ext}^\bullet_{U_q^{\mathrm{Lus}}(\mathfrak{n})}(\mathbb{C}, M)$, and
$\iota^*_{\check{\lambda}\cdot 0}(J_*^{\mathrm{Quant}}(M))$ is the $\check{\lambda}$-component of $\mathrm{Tor}^\bullet_{U_q^{\mathrm{KD}}(\mathfrak{n}^-)}(\mathbb{C}, M)$.

Lie Algebra Representation via Coherent Sheaves

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The infinite-dimensional theory bifurcates into the !-and the $*$-versions; here !-version is considered.

To each κ one can assign a *twisting* (an infinitesimal gerbe) on $\mathbb{B}G(K)^{\wedge}_{G(O)}$ and use it to twist the IndCoh category. A slight variant of above is

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Proposition ([Ras20])

When restricted to bounded-below objects, the functor

$$\mathsf{KL}_{\kappa}(B)_{\mathsf{ren}} \simeq \mathsf{IndCoh}^{!}_{\mathsf{ren},\kappa}(\mathbb{B}B(K)^{\wedge}_{B(O)}) \xrightarrow[\simeq]{\spadesuit} \mathsf{IndCoh}^{*}_{\mathsf{ren},\kappa-\kappa_{\mathsf{crit}}}(\mathbb{B}B(K)^{\wedge}_{B(O)})$$

$$\xrightarrow{*\text{-push}} \mathsf{IndCoh}^{*}_{\mathsf{ren},\kappa-\kappa_{\mathsf{crit}}}(\mathbb{B}T(K)^{\wedge}_{T(O)}) \simeq \mathsf{KL}_{\kappa-\kappa_{\mathsf{crit}}}(T)_{\mathsf{ren}}$$

coincides with Feigin's semi-infinite cohomology $C_*^{\frac{\infty}{2}}(\mathfrak{n}((t)), N(O), -)$.

Here κ_{crit} is the *critical* (a.k.a. *Tate*) shift, corresponding to $c = 0$. Existence of ($\spadesuit$) is a distinguished feature of the renormalized theory.

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- For every $(x, y) \in \mathbb{A}^2(\mathbb{C})$ where $x \neq y$, the corresponding base change gives $\mathrm{Vect} \otimes_{\mathrm{DMod}(\mathbb{A}^2)} \mathcal{KL}_\kappa(G)_{[2]} \simeq \mathrm{KL}_\kappa(G)_{\mathrm{ren}} \otimes \mathrm{KL}_\kappa(G)_{\mathrm{ren}}$;

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- For every $x \in \mathbb{A}^1(\mathbb{C})$ on the diagonal, the corresponding base change gives $\mathrm{Vect} \otimes_{\mathrm{DMod}(\mathbb{A}^2)} \mathcal{KL}_\kappa(G)_{[2]} \simeq \mathrm{KL}_\kappa(G)_{\mathrm{ren}}$;
- For every $(x, y) \in \mathbb{A}^2(\mathbb{C})$ where $x \neq y$, the corresponding base change gives $\mathrm{Vect} \otimes_{\mathrm{DMod}(\mathbb{A}^2)} \mathcal{KL}_\kappa(G)_{[2]} \simeq \mathrm{KL}_\kappa(G)_{\mathrm{ren}} \otimes \mathrm{KL}_\kappa(G)_{\mathrm{ren}}$;

The behavior as we approach the diagonal encodes the *fusion* structure of $\mathrm{KL}_\kappa(G)$.

Unitality means, for instance, that $\{x\} \hookrightarrow \{x, y\}$ yields a map

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given by $M \mapsto \mathbb{V}^0_\kappa \boxtimes M$, where

$$\mathbb{V}^0_\kappa := \mathrm{Ind}^{\mathrm{KL}_\kappa(G)^\heartsuit}_{\mathrm{Rep}(G(O))^\heartsuit}(\mathbb{C})$$

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It encodes the *fusion action* of $\mathrm{KL}_\kappa(G)$ on $\hat{\mathfrak{g}}_\kappa\text{-mod}^I$.

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Factorization modules internal to $\mathcal{IKL}_\kappa(G)$ are similarly defined (diagonal: M; off-diagonal: $A \boxtimes M$), and unitality gives

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For simplicity we write $C^{\frac{\infty}{2}} := C^{\frac{\infty}{2}}_*(\mathfrak{n}((t)), N(O), -)$. The map $\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \xrightarrow{\mathrm{Res}} \mathrm{KL}_\kappa(B)_{\mathrm{ren}} \xrightarrow{C^{\frac{\infty}{2}}} \mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}$ is a *lax-unital factorizable* functor, and thus factors through an "enhanced" map

$$C^{\frac{\infty}{2}}_{\mathrm{enh}} : \hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \to C^{\frac{\infty}{2}}(\mathbb{V}^0_\kappa)\text{-FactMod}(\mathcal{KL}_\kappa(T)).$$

Proposition ("Torus FLE")

There exists an equivalence of factorizable crystals of categories

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Problem: neither is easy to compute / explicitly known.

Matching Factorization Algebras

Proposition ([Gai21])

There exists an unique $\check{\Lambda}^{<0}$-graded factorization algebra Ω such that:

- *if $\check{\lambda} \notin \check{\Lambda}^{<0}$, then the !-fiber at $\check{\lambda}x$ is zero;*
- *the !-fiber at every $\check{\lambda}x$ has no negative cohomology;*
- *if $\check{\lambda}$ is a simple negative root, then either the $*$-fiber at $\check{\lambda}x$ is $\mathbb{C}[1]$, or the !-fiber at $\check{\lambda}x$ is $\mathbb{C}[-1]$;*
- *if $\check{\lambda}$ equals $w(\check{\rho}) - \check{\rho}$ for some $\ell(w) = 2$, then the !-fiber at $\check{\lambda}x$ vanishes at H^0 and H^1, and $*$-fiber at $\check{\lambda}x$ vanishes at H^0 and H^{-1};*
- *otherwise, the !-fiber at $\check{\lambda}x$ vanishes at H^0, and $*$-fiber at $\check{\lambda}x$ vanishes at H^0, H^{-1} and H^{-2}.*

One can use direct computation (using e.g. Kashiwara-Tanisaki localization) to verify this for both Ω^{KM} and Ω^{Quant}.

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Wakimoto modules are the $\frac{\infty}{2}$-analogues of Verma modules.

At generic c, $M_{\mathrm{KM}}^{!,\check{\lambda}}$ becomes the *affine Verma* module $\mathrm{Ind}_{\mathrm{Lie}(I)}^{\hat{\mathfrak{g}}_\kappa}(\mathbb{C})$, and $M_{\mathrm{KM}}^{*,\check{\lambda}}$ becomes the dual affine Verma module.

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- Placing another *costandard* object $M^{*,2\check{\rho}-\check{\mu}}_{\mathrm{KM}}$ at $\infty \in \mathbb{P}^1$;
- !-pushing $J^{\mathrm{KM}}_*(M^{!,\check{\lambda}}_{\mathrm{KM}}, M^{*,2\check{\rho}-\check{\mu}}_{\mathrm{KM}})_{0,\infty}|_{\mathrm{tot.deg.}=2\check{\rho}}$ along the Abel-Jacobi map $\mathrm{AJ} : \mathfrak{Conf}_{0,\infty} \to \mathrm{Bun}_{\check{T}}(\mathbb{P}^1)$;
- Pairing with the (pushforward of) dualizing sheaf of the $(\omega^{1/2}_{\mathbb{P}^1})^{2\check{\rho}}$-component of $\mathrm{Bun}_{\check{T}}(\mathbb{P}^1)$.

Localization

Set $\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty} := \mathrm{Bun}_G(\mathbb{P}^1) \times_{(\mathrm{pt}/G \times \mathrm{pt}/G)} (\mathrm{pt}/B \times \mathrm{pt}/B)$. There exists a *localization* functor

$$\mathrm{Loc}_G^{0,\infty} : \hat{\mathfrak{g}}_\kappa\text{-mod}^I \otimes \hat{\mathfrak{g}}_\kappa\text{-mod}^I \to \mathrm{DMod}_\kappa(\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty}),$$

where the !-fiber at the trivial bundle is given by conformal block of the two modules (placed at 0 and ∞) over $\mathbb{P}^1$.

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where the !-fiber at the trivial bundle is given by conformal block of the two modules (placed at 0 and ∞) over $\mathbb{P}^1$.

Work of N. Rozenblyum [Roz11] tells us that there is also a *chiral localization* functor

$$\mathrm{Loc}_{T,\Omega} : C^{\frac{\infty}{2}}(\mathbb{V}_\kappa^0)\text{-FactMod}(\mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}) \to \mathrm{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathrm{Bun}_T(\mathbb{P}^1));$$

the !-fiber is more interesting here (intuitively, it computes conformal block with $C^{\frac{\infty}{2}}(\mathbb{V}_\kappa^0)$ occupying everywhere away from $0, \infty$).

Let $\mathrm{CT}_* : \mathrm{DMod}_\kappa(\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty}) \to \mathrm{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathrm{Bun}_T(\mathbb{P}^1))$ denote the !-pull-$*$-push along

$$\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty} \leftarrow \mathrm{Bun}_B(\mathbb{P}^1) \rightarrow \mathrm{Bun}_T(\mathbb{P}^1)$$

(followed by a κ_{crit} shift).

Let $\mathsf{CT}_* : \mathsf{DMod}_\kappa(\mathsf{Bun}_G(\mathbb{P}^1)_{0,\infty}) \to \mathsf{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathsf{Bun}_T(\mathbb{P}^1))$ denote the !-pull-$*$-push along

$$\begin{array}{ccc} & \mathsf{Bun}_B(\mathbb{P}^1) & \\ \swarrow & & \searrow \\ \mathsf{Bun}_G(\mathbb{P}^1)_{0,\infty} & & \mathsf{Bun}_T(\mathbb{P}^1) \end{array}$$

(followed by a κ_{crit} shift).

The final piece of folklore that we prove is the commutativity of the following diagram:

$$\begin{array}{ccccc} \hat{\mathfrak{g}}_\kappa\text{-mod}^I & \xrightarrow{C^{\frac{\infty}{2}}} & C^{\frac{\infty}{2}}(\mathbb{V}^0_\kappa)\text{-FactMod}(\mathsf{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}) & \xrightarrow{\mathsf{FLE}_T} & \Omega^{\mathrm{KM}}\text{-FactMod}_{\mathrm{alg}} \\ \downarrow{\scriptstyle \mathsf{Loc}_G^{0,\infty}} & & \downarrow{\scriptstyle \mathsf{Loc}_{T,\Omega}} & & \downarrow{\scriptstyle \mathsf{AJ}_!} \\ \mathsf{DMod}_\kappa(\mathsf{Bun}_G(\mathbb{P}^1)_{0,\infty}) & \xrightarrow[\mathsf{CT}_*]{} & \mathsf{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathsf{Bun}_T(\mathbb{P}^1)) & \xrightarrow[\text{Fourier-Mukai}]{} & \mathsf{DMod}_{(\kappa-\kappa_{\mathrm{crit}})^{-1}}(\mathsf{Bun}_{\check{T}}(\mathbb{P}^1)) \end{array}$$

from which the $*$-fibers can be computed.

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