From braids to transverse slices in reductive groups

Wicher Malten[1]

[1]Mathematical Institute
University of Oxford
Wicher.Malten@maths.ox.ac.uk

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Outline

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- The fact that they are the semi-classical limits of finite W-algebras (and their affine cousins).
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- Appear in the work of numerous physicists on supersymmetric gauge theories (Gaoitto, Witten, etc.).

The Kostant Slice

- Kostant's slice: fixing a principal nilpotent element e in $\mathfrak{g}$, the Jacobson-Morozov theorem furnishes an embedding $\langle e, h, f \rangle = \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$; set $\mathfrak{s} := e + \ker \operatorname{ad} f \subset \mathfrak{g}$. It comes with two different cross section statements (from 1963 and 1978):

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- Denote by $N_+ = [B, B]$ the unipotent radical of a Borel subgroup B, by N_- its opposite.
- The adjoint action map

$$N_+ \times \mathfrak{s} \longrightarrow e + \mathfrak{n}_+^\perp =: \mu^{-1}(e)$$

  is an isomorphism, where $\mathfrak{n}_+^\perp = \mathfrak{b}_+$ denotes the Killing form complement to $\mathfrak{n}_+$.

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- For $w \in W$, write

$$N_w := N_+ \cap w^{-1} N_- w = \prod_{\beta \in \mathfrak{R}_w} N_\beta,$$

where $\mathfrak{R}_w$ is the set of positive roots made negative by w, and by T^w the points in T fixed by $\dot{w}$.

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- (Proof of the second cross section is missing!!)

The Steinberg Slice

Example

Let $G = \mathrm{SL}_{r+1}$ over a commutative ring $\mathcal{A}$ and consider the Coxeter element $w = s_1 \cdots s_r$. A suitable lift $\dot{w}$ yields the Steinberg slice of Frobenius companion matrices

$$\dot{w}N_w = \left\{ \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ (-1)^r & c_r & \cdots & c_2 & c_1 \end{bmatrix} : c_1, \dots, c_r \in \mathcal{A} \right\}.$$

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- In 2011, Sevostyanov constructed slices out of Weyl group elements whose "eigenspaces" in the reflection representation can be ordered "nicely" w.r.t. the dominant Weyl chamber.
- In 2012 (independently), He-Lusztig constructed slices of out *elliptic* Weyl group elements (= no fixed points in the reflection representation) which have minimal length.

An example

Example

Let $G = \mathrm{SL}_3$ over a commutative ring and $w := s_1 s_2 s_1$. The cross section statement asks whether the conjugation map

$$\left(\begin{bmatrix} 1 & n_1 & n_{12} \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x_1 & x_{12} & t \\ x_2 & -t^{-2} & 0 \\ t & 0 & 0 \end{bmatrix} \right) \in N_+ \times \dot{w} T^w N_+$$

to

$$\begin{bmatrix} n_{12}t + x_1 + n_1 x_2 & -n_1 t^{-2} + x_{12} - n_1(n_{12}t + x_1 + n_1 x_2) & n_1 n_2 t^{-2} + t - n_2 x_{12} + (n_1 n_2 - n_{12})(n_{12}t + x_1 + n_1 x_2) \\ n_2 t + x_2 & -t^{-2} - n_1(n_2 t + x_2) & n_2 t^{-2} + (n_1 n_2 - n_{12})(n_2 t + x_2) \\ t & -n_1 t & (n_1 n_2 - n_{12})t \end{bmatrix}$$

in $N_+ \dot{w} T^w N_+$ is an isomorphism.

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- Sevostyanov's 2019 computations show that in order to construct strictly transverse slices to *all* conjugacy classes in reductive groups, you need to use most non-elliptic classes.

The braid monoid: definition

- Weyl groups are examples of finite Coxeter groups, which have a presentation

$$W = \langle s_1, \ldots, s_{\mathrm{rk}} : s_i s_j s_i \cdots = s_j s_i s_j \cdots \ , \ s_i^2 = 1 \rangle_{\mathrm{grp}}$$

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- The corresponding braid *monoid* is given by

$$B^+ := B_W^+ := \langle b_1, \ldots, b_{\mathrm{rk}} : b_i b_j b_i \cdots = b_j b_i b_j \cdots \rangle_{\mathrm{mon}}$$

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- Moreover, any element in B can be expressed as a "fraction" of elements in B^+.
- Matsumoto's theorem furnishes a well-defined inclusion of sets

$$W \longrightarrow B^+, \qquad w \longmapsto b_w$$

by picking any *reduced* expression $w = s_{i_l} \cdots s_{i_1}$ and then mapping w to $b_{i_l} \cdots b_{i_1} =: b_{i_l \dots i_1} =: b_w$. The elements b_w are called *reduced/simple braids*.

The braid group: word problem

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$$(b_1 b_2 b_1 b_3 b_2 b_4)^3 \stackrel{?}{=} b_1 b_2 b_3 b_4 b_1 b_2 b_3 b_1 b_2 b_1 b_3 b_4 b_2 b_2 b_3 b_4 b_1 b_2.$$

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- Roughly speaking, the (right) Deligne-Garside normal form of a b braid in B^+ is obtained by decomposing it as a product of reduced braids $b = b_{w_n} \cdots b_{w_1}$, and then making the rightmost factors as large as possible.

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- We write $\mathrm{DG}(b) := \mathrm{DG}_1(b)$, and will often identify it with the corresponding Coxeter group element w_1.

Deligne-Garside normal form: back to examples

Example

Let W be of type A_2, now find

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Example

Let W be of type A_4, now find

$$\begin{aligned}(b_1 b_2 b_1 b_3 b_2 b_4)^3 &= b_{23} b_{341231} b_{w_\circ} \\ &= b_1 b_2 b_3 b_4 b_1 b_2 b_3 b_1 b_2 b_1 b_3 b_4 b_2 b_2 b_3 b_4 b_1 b_2.\end{aligned}$$

He-Lusztig's result

- Recall: Steinberg's claim is for Coxeter elements, e.g. $s_1 \cdots s_{\mathrm{rk}}$ where rk is the rank of W (or G): the conjugation action

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- The cross sections of He-Lusztig apply to elliptic elements w of minimal length in their conjugacy class, in the same way:

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- (1): Proven "directly" for all elements w, such that $\mathrm{DG}(b_w^d) = w_\circ$ for some integer $d \geq 1$. From case-by-case work (Geck-Michel), it was known then that this is true for *some* elements of minimal length in each elliptic conjugacy class, when $d = \mathrm{ord}(w)$.

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- (2): If it is true for an element $w = xy$ with $\ell(w) = \ell(x) + \ell(y)$, then it is also true for $w' := yx$ if $\ell(y) + \ell(x) = \ell(w')$. From case-by-case work (Geck-Pfeiffer), it was known then that all elliptic elements of minimal length are conjugate to each other by such *cyclic shifts*.

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- (2'): Simpler: if w and w' are conjugate by cyclic shifts and $\mathrm{DG}(b_w^d) = w_\circ$, then $\mathrm{DG}(b_{w'}^{d'}) = w_\circ$ for some d'.

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Lemma

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Lemma

Sevostyanov's elliptic elements satisfy this braid equation.

- Do his non-elliptic satisfy it? Rarely... but those slices are a bit different!

New definitions: firmly convex elements

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Lemma

$\mathfrak{R}^w$ forms a standard parabolic subsystem if and only if the complement $\mathfrak{R}_+ \backslash \mathfrak{R}^w$ is convex, i.e.:
If $\beta_0, \beta_1 \in \mathfrak{R}_+ \backslash \mathfrak{R}^w$ and $c_0, c_1 \in \mathbb{R}_{>0}$ are such that $c_0\beta_0 + c_1\beta_1$ is again a root, then it lies in $\mathfrak{R}_+ \backslash \mathfrak{R}^w$.

New definitions: braid power bound

Definition

Let $w_\circ$ denote the longest element of W. Given a firmly convex element w, let w_f denote the longest element of the standard parabolic subsystem $\mathfrak{R}^w$; this yields a *braid power bound*

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- So $\mathfrak{R}_{w_\circ w_f} = \mathfrak{R}_+ \backslash \mathfrak{R}^w$.

New definitions: dominant elements

Definition

Let C denote the dominant Weyl chamber. For any w, let $V_w = \mathrm{im}(\mathrm{id} - w)$ denote the orthogonal complement to the subset of fixed points $\ker(\mathrm{id} - w)$ in the reflection representation.

New definitions: dominant elements

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Let C denote the dominant Weyl chamber. For any w, let $V_w = \mathrm{im}(\mathrm{id} - w)$ denote the orthogonal complement to the subset of fixed points $\ker(\mathrm{id} - w)$ in the reflection representation. Then w is called *dominant* if the closure $\overline{C}$ of C contains an open subset of V_w.

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Example

Reflection in a root is dominant if and only if this root is the highest root or the highest short root.

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For any element w there are implications

$$\textit{elliptic or Sevostyanov element} \Longrightarrow \textit{dominant} \Longrightarrow \textit{firmly convex}$$

Transversality

- Let G be a manifold (or variety), and let C and S be two submanifolds. We say that the intersection $C \cap S$ is *transverse* if for all $g \in C \cap S$, we have

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- We say that the intersection is *strictly transverse* if this is a direct sum, i.e.

$$T_g C \cap T_g S = \{0\}.$$

Inspiration from braids

- For any w, analysing roots shows that

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- By the "convexity" lemma, this inclusion is strict if w is not firmly convex; if it is firmly convex then it is equivalent to

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in the left weak Bruhat-Chevalley order.

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  in the left weak Bruhat-Chevalley order.
- We can modify Sevostyanov's definitions to come up with a cross section *statement*

$$N \times \dot{w} L^w N_w \longrightarrow N \dot{w} L^w N,$$

  for any firmly convex element w. Here $N \subseteq N_+$ is generated by root subgroups for roots in $\mathfrak{R}_+ \backslash \mathfrak{R}^w$, whereas L^w is the reductive subgroup "generated" by $\mathfrak{R}^w$ and T^w.

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Theorem

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then the conjugation map

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is an isomorphism, over any commutative ring.

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Lemma

He-Lusztig's and Sevostyanov's elements satisfy this equation.

More?

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- How about the converse?

Poisson structures

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Lemma

This Poisson bracket reduces to a Poisson bracket on the slices if and only if such a twist is made.

Transversality again

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Example

Consider $w = s_3 s_1 s_2 s_3$ in type B_3; it does not fix any roots so it is closed, but for any integer $d > 1$ we have

$$\mathrm{DGN}(b_w^d) = b_w^d \qquad \text{and} \qquad \mathrm{DGN}(b_{w^{-1}}^d) = b_{323} b_w^{d-2} b_{13213}.$$

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- We will see that this is true, with $d' = d$. Surprising... because normally $\mathrm{DG}(b_w^d)$ and $\mathrm{DG}(b_{w^{-1}}^d)$ are very different!

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- The cross section statement is almost a statement about roots.
- But what is the identity $\mathrm{DG}(b_w^d) = w_\circ w_f$ *really* doing in the proof?
- It's trying to make all the roots in $\mathfrak{R}_+ \backslash \mathfrak{R}^w$ negative, step by step:

$$\begin{aligned} \mathrm{DG}(b_w^d) = w_\circ w_f \quad &\Longrightarrow \quad \mathrm{cross}_w^d(\mathfrak{R}_+ \backslash \mathfrak{R}^w) = \varnothing \\ &\Longrightarrow \quad \text{cross section is isomorphism} \end{aligned}$$

Crossing roots

Definition

For any positive root β and w, we obtain a subset of positive roots

$$\mathrm{cross}_w(\beta) := \{w(\beta + \sum_{i=1}^{m} \beta_i) \in \mathfrak{R} : \beta_1, \ldots, \beta_m \in \mathfrak{R}_w, m \geq 0\} \cap \mathfrak{R}_+$$

and for a subset of positive roots $\mathfrak{N} \subseteq \mathfrak{R}_+$ we set

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Example

What is $\mathrm{cross}_w(\beta)$ when β lies in $\mathfrak{R}_w$? When $w(\beta)$ is simple?

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- *Implies: For any other element* v *of* W *and integer* $d \geq 0$,

$$\mathrm{DG}(b_w^d) \geq v \qquad \textit{if and only if} \qquad \mathrm{cross}_w^d(\mathfrak{R}_v) = \varnothing,$$

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- *In particular:* w *is firmly convex and satisfies the braid equation* $\mathrm{DG}(b_w^d) = w_\circ w_f$ *if and only if* $\mathrm{cross}_w^d(\mathfrak{R}_+ \backslash \mathfrak{R}^w) = \varnothing$.

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- *In particular:* w *is firmly convex and satisfies the braid equation* $\mathrm{DG}(b_w^d) = w_\circ w_f$ *if and only if* $\mathrm{cross}_w^d(\mathfrak{R}_+ \backslash \mathfrak{R}^w) = \varnothing$.
- *And that easily implies: if* w *is firmly convex then* $\mathrm{DG}(b_w^d) = w_\circ w_f$ *if and only if* $\mathrm{DG}(b_{w^{-1}}^d) = w_\circ w_f$.

Strict transversality: minimally dominant elements

Definitions

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Example

For elliptic conjugacy classes, "minimally dominant" = "has minimal length".

Lemma

For (nontrivial) non-elliptic conjugacy classes, minimally dominant elements never have minimal length.

Braid powers of minimally dominant elements

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- Combine: $\Rightarrow$ they all satisfy $\mathrm{DG}(b_w^d) = w_\circ w_f$ for some d
- So by the previous theorem, they all yield transverse slices!

Strict transversality and minimally dominant elements

- In 2019, Sevostyanov showed that a subset of his elements yield strictly transverse slices (by using Lusztig's partition).

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- He already knew they were transverse, so his main ingredient is a case-by-case dimension calculation.
- Can show that these elements are all minimally dominant.
- Can now deduce that all minimally dominant elements in these conjugacy classes yield strictly transverse slices!

Final statement

Theorem

Let C be a conjugacy class of a connected reductive group over an algebraically closed field, and let w be a minimally dominant element in the corresponding conjugacy class in Lusztig's partition.

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Then C is strictly transversally intersected by $\dot{w}L^{w}N_{w}$, and this slice inherits a natural Poisson structure.

End

- Thanks for listening!!
- Questions? Ideas??
- w.malten@gmail.com