Structure of Harish-Chandra cells

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Outline

Introduction

Kazhdan-Lusztig cells

Harish-Chandra cells

Real forms and counting cells

Slides at `http://www-math.mit.edu/~dav/paper.html`

What's a Harish-Chandra cell?

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R}) = G(\mathbb{R})^\theta$

$G \supset K = G^\theta$ complexifications, $\mathfrak{g} = \mathrm{Lie}(G)$

Cartan and Borel $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$

$\lambda \in \mathfrak{h}^*$ dom reg, $\mathcal{M}(\mathfrak{g}, K)_\lambda =$ $(\mathfrak{g}, K)$-mods of infl char λ

$\mathrm{Irr}(\mathfrak{g}, K)_\lambda =$ irr reps, $K\mathcal{M}(\mathfrak{g}, K)_\lambda = \mathbb{Z} \cdot \mathrm{Irr}(\mathfrak{g}, K)_\lambda$ Groth grp.

Integral Weyl group $W(\lambda)$ acts on $K\mathcal{M}(\mathfrak{g}, K)_\lambda$; $\leftrightsquigarrow$ left reg rep of W studied by Kazhdan-Lusztig.

Preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$: Kazhdan-Lusztig def is

$$Y \underset{LR}{\leq} X \iff \exists w \in W(\lambda),\ [Y] \text{ appears in } w \cdot X.$$

Rep-theoretic def is (with F fin-diml rep of G^{ad})

$$Y \underset{LR}{\leq} X \iff \exists F, Y \text{ comp factor of } F \otimes X.$$

Equiv rel $Y \underset{LR}{\sim} X$ means $Y \underset{LR}{\leq} X \underset{LR}{\leq} Y$; complement is $Y \underset{LR}{<} X$.

A Harish-Chandra cell is an $\underset{LR}{\sim}$ equiv class in $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

What's true about Harish-Chandra cells?

Theorem. (Consequence of rep theory defn of cells.)

1. $Y \underset{LR}{\leq} X \implies \mathcal{AV}(Y) \subset \mathcal{AV}(X)$.
2. $Y \underset{LR}{<} X \implies \mathcal{AV}(Y) \subsetneq \mathcal{AV}(X)$.
3. $Y \underset{LR}{\sim} X \implies \mathcal{AV}(Y) = \mathcal{AV}(X)$.

$C(X) =_{\underset{LR}{\sim}}$ equiv class of X = HC cell $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\overline{C}(X) =_{\underset{LR}{\leq}}$ interval below X = HC cone $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\partial C(X) = \overline{C}(X) - C(X)$.

Theorem. (Consequence of KL defn of cells).

1. $W(\lambda)$ acts on $\overline{C}_{\mathbb{Z}}(X) = \left[\sum_{Y \underset{LR}{\leq} X} \mathbb{Z}Y\right] \supset \partial C_{\mathbb{Z}}(X)$.
2. $W(\lambda)$ acts on $C_{\mathbb{Z}}(X) \simeq \overline{C}_{\mathbb{Z}}(X)/\partial C_{\mathbb{Z}}(X)$.
3. $C_{\mathbb{Z}}(X)$ contains unique special rep $\sigma(X) \in \widehat{W(\lambda)}$.
4. $\mathcal{AV}(X) =$ union of closures of K-forms of $\mathcal{O}(\sigma(X))$.

Cplx nilp orbit $\mathcal{O}(\sigma(X))$ def by Springer corr.

KL cells

Theorem (Kazhdan-Lusztig)

1. KL relations $\underset{L}{\sim}$ and $\underset{LR}{\sim}$ partition W into left cells and two-sided cells $C_L(w) \subset C_{LR}(w)$ $(w \in W)$.
2. $\mathbb{Z}$-module $C_{\mathbb{Z},L}(w)$ carries a rep of W.
3. $C_{\mathbb{Z},LR}(w)$ carries rep of $W \times W$.
4. $\sum_{C_{LR}} C_{\mathbb{Z},LR} \simeq \mathbb{Z}W$, regular representation of W.
5. Two-sided cells C_{LR} partition $\widehat{W}$ into subsets $\Sigma(C_{LR})$ called families: $C_{\mathbb{Z},LR} \simeq \sum_{\sigma \in \Sigma(C_{LR})} \sigma \otimes \sigma^*$.
6. As rep of the first W, $C_{\mathbb{Z},LR} \simeq \sum_{C_L \subset C_{LR}} C_{\mathbb{Z},L}$.

Lusztig's description of families

For any finite group F, Lusztig in 1979 defined

$$\mathcal{M}(F) = \{\,(x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\,\}/(\text{conjugation by } F)$$

The group F acts itself by conjugation;

$$\mathcal{M}(F) \simeq \text{irr } F\text{-eqvt coherent sheaves on } F.$$

Theorem (Lusztig) Suppose that Σ is a family in $\widehat{W}$.

1. Σ has one special representation $\sigma_s(\Sigma) \in \widehat{W}$.
2. $\sigma_s \underset{\text{Springer}}{\longleftrightarrow}$ special nilpotent orbit $\mathcal{O}_s(\Sigma) = \mathcal{O}_s(\sigma_s) \subset \mathcal{N}^*/G$.
3. Write $A(\mathcal{O}_s) = \pi_1^G(\mathcal{O}_s)$ (eqvt fund grp). Write

$$\{\sigma_s = \sigma_1, \sigma_2 \ldots, \sigma_r)\} = \Sigma \cap (\text{Springer}(\mathcal{O}_s))$$

all W-reps in $\Sigma \underset{\text{Springer}}{\longleftrightarrow} \xi_j \in \widehat{A(\mathcal{O}_s)}$. Define

$$\overline{A} = \overline{A}(\mathcal{O}_s) = A(\mathcal{O}_s)/[\cap_j \ker \xi_j]$$

4. Have inclusion $\Sigma \hookrightarrow \mathcal{M}(\overline{A})$, $\quad \sigma \mapsto (x(\sigma), \xi(\sigma))$ so

$$x(\sigma_s) = x(\sigma_j) = 1 \in \overline{A}, \quad \xi(\sigma_j) = \xi_j \in \widehat{\overline{A}}.$$

Lusztig's description of left cells

Recall that finite group F gives

$$\mathcal{M}(F) = \{(x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\}/(\text{conj by } F$$

$$\simeq \text{irr conj-eqvt coherent sheaves } \mathcal{E}(x,\xi) \text{ on } F.$$

Given subgroup $S \subset F$, const sheaf $\mathcal{S}$ on S is S-eqvt for conj. Push forward to F-eqvt sheaf supp on F-conjs of S:

$$i_*(\mathcal{S}) = \sum_{s,\xi} m_S(s,\xi)\mathcal{E}(s,\xi), \qquad m_S(s,\xi) = \dim \xi^{S^s}.$$

Sum runs over S conj classes $s \in S$. Can write this as

$$i_*(\mathcal{S}) = \sum_s \mathcal{E}(s, \mathrm{Ind}_{S^s}^{F^s}(\text{triv})).$$

Theorem (Lusztig) $C_L \subset C_{LR} \leftrightarrow \Sigma \subset \widehat{W}$, $\quad \overline{A}$ fin grp,

$$\Sigma \hookrightarrow \mathcal{M}(\overline{A}), \quad \sigma \mapsto (x(\sigma), \xi(\sigma)).$$

1. $\exists$ subgp $\Gamma = \Gamma(C_L) \subset \overline{A}$ so $C_{\mathbb{Z},L} \simeq \sum_{x,\xi} m_\Gamma(x,\xi)\sigma(x,\xi)$
2. $m_\Gamma(1,\text{triv}) = 1$, so special rep σ_s appears once in $C_{\mathbb{Z},L}$.
3. $\exists$ Lusztig left cells with $\Gamma = \overline{A}$, so $C_{\mathbb{Z},L} \simeq \sum_x \sigma(x,\text{triv})$.
4. G classical $\implies$ $\exists$ Springer left cells with $\Gamma = \{e\}$, so $C_{\mathbb{Z},L} \simeq \sum_{\xi \in \widehat{\overline{A}}} \dim(\xi)\sigma(1,\xi)$, Springer reps for $\mathcal{O}_s$ in Σ.

Consequences of Lusztig for HC cells

HC world $(\mathfrak{g}, K)$: $\operatorname{Irr}(\mathfrak{g}, K)_\lambda \supset C =$ HC cell $\rightsquigarrow W(\lambda)$ rep $C_{\mathbb{Z}}$.

$C_{\mathbb{Z}} \supset \sigma_s(C)$ special in $\widehat{W(\lambda)} \rightsquigarrow \mathcal{O}(\mathcal{C})$, $\quad \Sigma(\mathcal{O}) \subset \widehat{W}$, $\quad \overline{A}(\mathcal{O})$ finite.

Theorem (McGovern, Binegar) $\mathcal{C}$ a HC cell in $\operatorname{Irr}(\mathfrak{g}, K)_\lambda$ as above.

1. $C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma$, $\quad m_C(\sigma) \in \mathbb{N}$, $\quad m_C(\sigma_s) = 1$.
2. $G(\mathbb{R})$ real form of type A, $SO(n)$, $Sp(2n)$, or exceptional, $\implies \exists S(C) \subset \overline{A}$ so $m_C(\sigma(x,\xi)) = m_{S(C)}(x,\xi)$.
3. $G(\mathbb{R})$ cplx, so $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_1$, $\overline{A}(\mathcal{O}) = \overline{A_1} \times \overline{A_1}$, then $S(C) = (\overline{A_1})_\Delta$, not one of Lusztig's Γ unless $A_1 = 1$.
4. In all other cases of (2), $S(C)$ is one of Lusztig's subgroups Γ from description of left cells.

(McGovern) (4) fails for some forms of $Spin(n)$, $PSp(2n)$.

Conjecture. Part (2) is true for any HC cell C.

Next goal: relate cells to real forms of orbit, try to prove conjecture in this way.

Real forms of G

Pinning $\mathcal{P} = (G, B, H, \{X_\alpha \mid \alpha \in \Pi(B, H) \subset X^*(H)\})$.

Langlands dual $({}^\vee G, {}^\vee B, {}^\vee H, \{X_{{}^\vee\alpha} \mid {}^\vee\alpha \in \Pi({}^\vee B, {}^\vee T)\})$.

dual pinning ${}^\vee\mathcal{P} = ({}^\vee G, {}^\vee B, {}^\vee H, \{X_{{}^\vee\alpha})\})$.

distinguished invs $\delta \in \mathrm{Aut}(\mathcal{P})$, neg transpose ${}^\vee\delta \in \mathrm{Aut}({}^\vee\mathcal{P})$.

Extended groups $G^\Gamma = G \rtimes \{1, \delta\}, \quad {}^\vee G^\Gamma = {}^\vee G \rtimes \{1, {}^\vee\delta\}$.

Here $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$; ${}^\vee G^\Gamma$ is Galois form of ${}^L G$.

Strong real form of G = G-conj class of $x \in G\delta$, $x^2 \in Z(G)$.

Strong form $x \rightsquigarrow$ inv aut $\theta_x = \mathrm{Ad}(x)$ Cartan for real form.

Summary: (conj classes of invs in G) $\leftrightsquigarrow$ ($\mathbb{R}$-forms of G).

Ex: $G = GL(n)$, involution $x_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \leftrightsquigarrow U(p, q)$.

Coming up: (involution respecting ??) $\leftrightsquigarrow$ (real form of ??).

Real forms of nilpotents

Pinning $\mathcal{P} = (G, B, H, \{X_\alpha\})$.

θ Cartan inv $\rightsquigarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ $\quad \mathcal{N} =$ nilp cone $\supset \mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{s}$.

Theorem (Jacobson-Morozov, Kostant, Kostant-Rallis)
$\mathcal{O} \subset \mathfrak{g}$ nilpotent orbit.

1. $\exists$ Lie triple $(T, E, F), [T, E] = 2E, [T, F] = -2F, [E, F] = T$,

$$E \in \mathcal{O}; \quad T \in \mathfrak{h} \text{ dom}; \quad T \text{ is unique} \rightsquigarrow \phi\colon SL(2) \to G$$

Define $\mathfrak{g}[j] = \{X \in \mathfrak{g} \mid [T, X] = jX\}$. JM parabolic is

$$\mathfrak{l} = \mathfrak{g}[0], \quad \mathfrak{u} = \sum_{j>0} \mathfrak{g}[j], \quad \mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

2. $G^E = (L^E)(U^E) = G^\phi U^F$ Levi decomp.
3. BIJECTION ($\mathbb{R}$-forms of $\mathcal{O}$) $\rightsquigarrow$ (G^ϕ-conj classes
   $\{\ell \in G^\phi \mid \ell^2 \in \phi(-I)Z(G)\}$, $\ell \mapsto \mathbb{R}$-form $x = \ell \cdot \phi\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\}$).
4. Summary: (conj classes of invs in G^ϕ) $\leftrightsquigarrow$($\mathbb{R}$-forms of $\mathcal{O}$).

Nilpotents in $U(p, q)$

Not presented in seminar.

Ex: $G = GL(n)$, $\mathcal{O} \leftrightsquigarrow$ partition $n = [m_1^{r_1}, \dots, m_k^{r_k}]$

Here $m_1 > \cdots > m_k$, $r_i > 0$.

$G^\phi = GL(r_1) \times \cdots \times GL(r_k)$.

Prev slide: (conj classes of invs in G^ϕ) $\leftrightsquigarrow$($\mathbb{R}$-forms of $\mathcal{O}$).

Conj classes of relevant invs in G^ϕ: write $r_j = p_j + q_j$,

$$\ell_j = i^{m_j - 1} \begin{pmatrix} I_{p_j} & 0 \\ 0 & -I_{q_j} \end{pmatrix}$$

Theorem (classical)

1. $\mathbb{R}$-forms of $\mathcal{O}$ in (equal rank) $GL(n)$ $\leftrightsquigarrow$ $[(p_j, q_j)]$.
2. The real nilpotent $\mathcal{O}([(p_j, q_j)])$ is in $U(p, q)$, where

$$p = \sum_j p_j[(m_j+1)/2] + q_j[m_j/2], \quad q = \sum_j p_j[m_j/2] + q_j[(m_j+1)/2].$$

3. (What looks like) "natural most split form" $\mathcal{O}([r_j, 0])$ is in $U(\sum r_j[(m_j + 1)/2], \sum r_j[m_j/2])$, which fails to be quasisplit as soom as partition has at least three odd parts.

Relation to cells

Vague conjs below are serious correction of statements in seminar.

Recall $\mathcal{O}$ nilpotent $\rightsquigarrow \phi\colon SL(2) \to G$, $A(\mathcal{O}) = G^\phi/G^\phi_0$.

Recall HC cell $\mathcal{C} \rightsquigarrow$ (one or more) $\mathbb{R}$-forms of nilp $\mathcal{O}$

$\rightsquigarrow$ (one or more) $\ell_i \in G^\phi$ involutions

$\rightsquigarrow$ (one or more) symm subgps $K_i^\phi \subset G^\phi$

$\rightsquigarrow$ (one or more) subgps $\overline{A}_i = \mathrm{im}(K_i^\phi) \subset \overline{A}(\mathcal{O}) = G^\phi/?$

CONJECTURE (Cells attached to $\mathcal{O}$) $\approx$ invs $\ell_i \in G^\phi$

CONJECTURE CONTINUED $S(C(\ell)) \approx \overline{A}_\ell$

Theorem (Barbasch-Vogan) Harish-Chandra cells in "big block" of reps of $U(p,q)$ are indexed (by $\mathcal{AV}$) by real forms of nilpotent orbits. In particular, $\mathcal{AV}(X) =$ closure of one nilpotent K-orbit.

To make conjs precise, thm general, look also at ${}^\vee G$. . .