2d TQFT's and partial fractions

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MIT Lie Groups Seminar

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(joint with M.Khovanov and
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d-dim'l

<u>Definition</u> (M.Atiyah): A TQFT is a symmetric tensor functor Cob_d→Vec. ← make more general

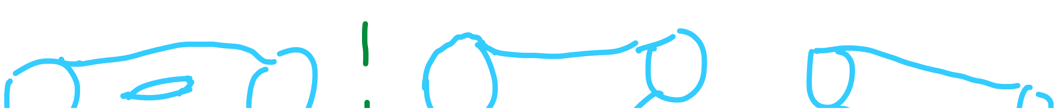
Generalization: replace Vec by a symmetric tensor category C
C-valued TQFT: symmetric tensor functor Cob_d→C.

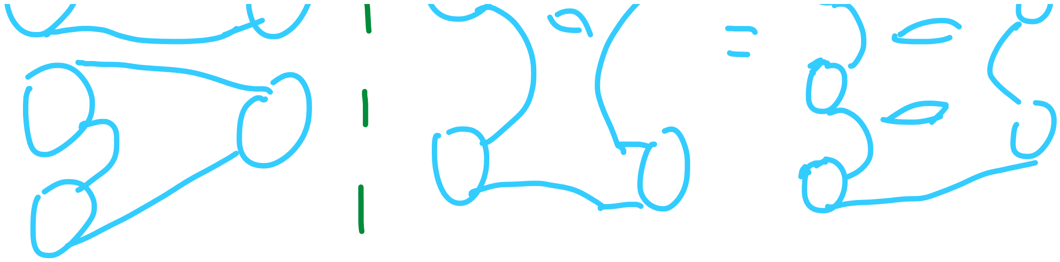
Examples: C=Vec, sVec, Rep(G) etc

Today: d=2!

Category of cobordisms Cob_d: d=2 O O O

Objects: (d-1)-dimensional closed oriented manifolds
Morphisms: d-dimensional oriented cobordisms
Composition: gluing

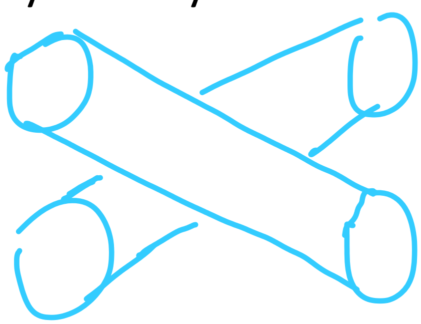




Identity morphisms=?

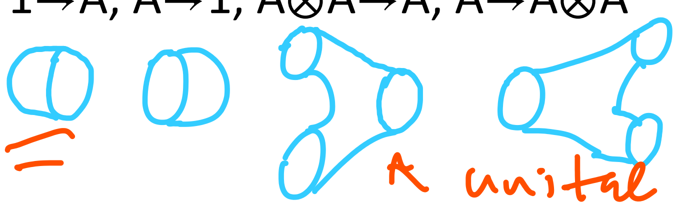
Tensor product: disjoint union
Unit object=?

Symmetry:



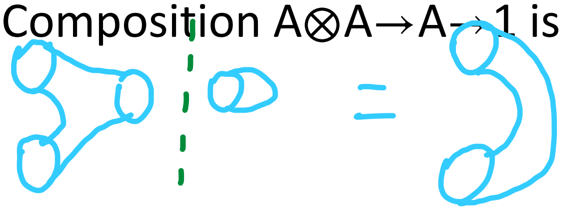
Some important objects of Cob_2:
empty set=1 and circle=A

Some important morphisms:
1→A, A→1, A⊗A→A, A→A⊗A



unital
commutative associative

Composition A⊗A→A→1 is non-degenerate



$A \simeq A^*$

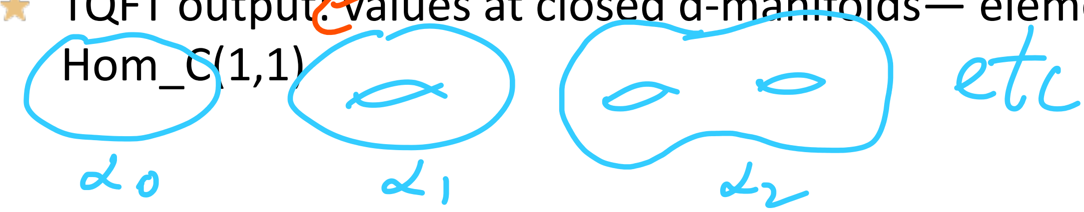
⭐ Thus A is a commutative Frobenius object in Cob_2:

commutative associative unital monoid equipped with a map A→1 such that the composition A⊗A→A→1 is a non-degenerate pairing.

<u>Theorem</u> (R.Dijkgraaf + folklore): Cob_2 is free category generated by the commutative def Frobenius object A.

<u>Corollary</u>: C-valued 2d TQFT's = Functors Cob_2→C = commutative Frobenius objects in C.

★ TQFT output: values at closed d-manifolds— elements of Hom_C(1,1)



Linear setup: choose a field k
C — k-linear category
Hom_C(1,1)=k

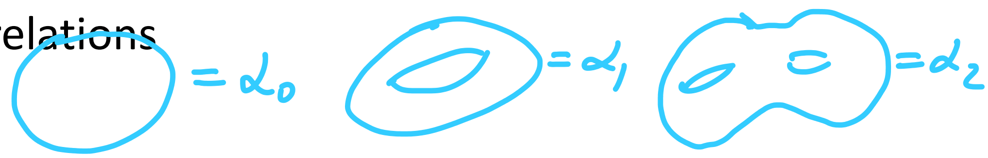
Frobenius object = Frobenius algebra

TQFT output: sequence $\alpha_0, \ \alpha_1, \alpha_2$, …. of elements of k

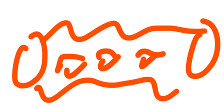
★ <u>Main Question:</u> Which sequences we will observe?

<u>Answer 1</u>: all sequences appear.

Take C = linearized Cob_2 modulo relations



etc



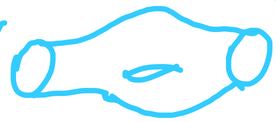
Hom's are ∞ dim'l

- Realization of sequence $\alpha = (\alpha_0,\ \alpha_1, \alpha_2, \ldots.)$: pair (C,A) such that the corresponding TQFT outputs α.

  Realization is finite if Hom spaces in C are finite dimensional.

  "eventually recurrent"

  <u>Answer 2</u> (M.Khovanov): sequence α admits a finite realization if and only if it is ~~linearly recursive~~, that is the generating function Z(T)= $\alpha_0 + \ \alpha_1 T + \alpha_2 T^2 + \ldots$ is rational.

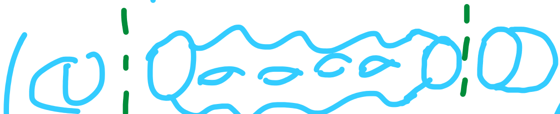
- $x = F($  $) \in \mathrm{Hom}_C(A,A)$

  $1, x, x^2, x^3, \ldots$ are linearly dependent

  e.g. $x^3 - 2x^4 + 3x^5 = 0$

  hence $x^n - 2x^{n+1} + 3x^{n+2} = 0$ for $n \geq 3$

  hence $\alpha_n - 2\alpha_{n+1} + 3\alpha_{n+2} = 0$ for $n \geq 3$

  $\alpha_n = F($  $) =$

  $F($  $)\, x^n\, F($  $)$

  Khovanov–Sazdanovic

  Existence: some quotients ~~Cob_2(α)~~ of linearized Cob_2.

- Realization is abelian if C is abelian, rigid, with finite dimensional Hom spaces.

  <u>Answer 3</u> (M.Khovanov, Y.Kononov, V.O.): sequence α admits an abelian realization if and only if it is

1) linearly recursive, Z(T)=$\dfrac{P(T)}{Q(T)}$ with relatively prime P(T) and Q(T).

$\bar{k}$

2) Q(T) has no multiple roots (in K) and deg P $\leq$ deg Q+1.

Thus Z(T)=$\delta_0 + \delta_1$T+ $\sum_i \dfrac{\beta_i}{1-\gamma_i T}$

↑ and in char k=0

3) If char k=p>0 then δ_1 and $\beta_i\gamma_i$ (for all i) are in the prime subfield $F_p \subset$ k

Remark: conditions 2) and 3) can be expressed in terms of 1-form Z(T)$\dfrac{dT}{T^2}$:

2. all its poles are simple except, possibly, at T=0
3. All its residues are in F_p

Examples:

1) α=1,1,1,1,… Thus Z(T)=$\dfrac{1}{1-T}$ and α has an abelian realization over any field.

C = Vec

A = K, A→K id

2) α=1,2,3,4,5,… Thus Z(T)=$\dfrac{1}{(1-T)^2}$ and α has no abelian realization over any field

3) α=1,2,1,2,1,2,… Thus Z(T)=$\dfrac{1+2T}{1-T^2} = \dfrac{\frac{3}{2}}{1-T} + \dfrac{-\frac{1}{2}}{1+T}$, so α has an abelian realization over any field of characteristic not 2.

4) α=1,1,2,3,5,8,13,… Thus Z(T)=$\dfrac{1}{1-T-T^2}$

α has an abelian realization: p=0,11,19,29,31,41,

α has no abelian realization: p=2,3,5,7,13,17,23,37

Note that: α=-1,2,1,3,4,7,11,… has an abelian realization over any field.

Remark: abelian realization in characteristic p>0 implies a realization with C=Vec. This is NOT the case in characteristic 0. Deligne categories (like Rep(S_t)) of super-exponential growth are needed.

Example:

1) α=1,2,1,2,1,2,… so Z(T)=$\frac{\frac{3}{2}}{1-T}+\frac{-\frac{1}{2}}{1+T}$; realization of α requires Rep(S_t) with t=3/2 and t=1/2.
2) α=3,1,3,1,3,1,… Here Z(T)=$\frac{2}{1-T}+\frac{1}{1+T}$; realization of α requires Rep(S_{-1}).

<u>Answer 4</u> (well known?) Assume p=0 and Z(T)=$\delta_0+\delta_1$T $+\sum_i \frac{\beta_i}{1-\gamma_i T}$

Sequence α admits an abelian realization of exponential growth if and only if δ_1 is an integer and $\beta_i\gamma_i$ (for all i) are integers >0.

This implies a realization with C=sVec.

Sequence α admits an realization with C=Vec if and only if δ_1 is an integer >1 and $\beta_i\gamma_i$ (for all i) are integers >0.

- Crucial tool: quotients by negligible morphisms ("gligible" quotients) aka "semisimplifications".

  Theorem of Andre-Kahn (abstract version of Jannsen theorem) implies:

  <u>Corollary</u>: α admits an abelian realization if and only if the gligible quotient of Cob_2(α) is semisimple.

  Important computation: compute semisimplifications of Cob_2(α).

- Results (k algebraically closed):

  1) if Z(T)=sum of partial fractions then the gligible quotient is a product of quotients for each summand.

  Thus if Z(T)=$\delta_0 + \delta_1$T+ $\sum_i \frac{\beta_i}{1-\gamma_i T}$ we need to consider only the following cases:

  2) If Z(T)=$\frac{\beta}{1-\gamma T}$ then the semisimplification of Cob_2(α) is semisimplification of Rep(S_t) with t=$\beta\gamma$ (exceptional values of t: non-negative integers)
  3) if Z(T)=$\delta_0 + \delta_1$T with $\delta_1 \neq 0$ then the semisimplification of Cob_2(α) is semisimplification of Rep(O_t) with t=$\delta_1 - 2$ (exceptional values of t: integers)
  4) if Z(T)=δ_0 then the semisimplification of Cob_2(α) is Rep(osp(1|2)) or, in characteristic p, a semisimplification

Example: Z(T)=1+2T. In characteristic $\neq 2$ we get C=Vec and A=k$[x]/x^2$. In characteristic 2 we get C=Z/2Z-graded vector spaces and A=k$[x]/(x^2-1)$. $(x-1)^2$

<u>Exercise</u>: Repeat the story above for d=1.

<u>Exercise</u>*: Same with d=3.