Coherent Springer theory and categorical Deligne-Langlands

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Overview

1. Context: (finite) Springer theory
2. Affine Springer theory
3. Coherent Springer theory
4. Methods: traces and Hochschild homology
5. Connections with Deligne-Langlands

Representations of Chevalley groups

G reductive group

Problem

Classify irreducible representations of $G(\mathbb{F}_q)$, compute characters

Strategy

Parabolic induction e.g. Borel $B \subset G$, Levi quotient $H = B/U$

Reps and chars: $H(\mathbb{F}_q) \xrightarrow{\text{inflate}} B(\mathbb{F}_q) \xrightarrow{\text{induce}} G(\mathbb{F}_q)$

Problem

Decompose $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\mathbb{C}$ and characters

Character sheaves

Idea: categorify characters and parabolic induction

$$\mathrm{Fun}(G/G) \rightsquigarrow \mathrm{Sh}(G/G)$$

To get geometry: $k = \mathbb{C}$ or $k = \overline{\mathbb{F}_q}$

Parabolic induction

$$G/G \xleftarrow{\mu} B/B \xrightarrow{\nu} H/H$$

- $\mathbb{F}_q \subset k = \overline{\mathbb{F}_q}$, then $X/\mathbb{F}_q \rightsquigarrow \mathrm{Fr} \curvearrowright X/k$
- Fr-equivariant sheaf $\mathcal{F} \rightsquigarrow \mathrm{tr}(\mathsf{Fr}^*, \mathcal{F}_x) \in \mathrm{Fun}(X^{\mathsf{Fr}}) = \mathrm{Fun}(X(\mathbb{F}_q))$
- Functorial: recover character of $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\mathbb{C})$, decomposition

Springer theory

Parabolic induction

$$\mathcal{N}/G \xleftarrow{\mu} \mathfrak{n}/B = \widetilde{\mathcal{N}}/G \xrightarrow{\nu} \{0\}/H$$

$$\text{Springer resolution: } \mu : \widetilde{\mathcal{N}} = G \times^B \mathfrak{n} \to \mathcal{N}$$
$$(g, n) \mapsto gng^{-1}, \qquad (gb, n) \sim (g, b^{-1}nb)$$
$$\widetilde{\mathcal{N}} \simeq \{(x, B) \in \mathcal{N} \times \mathcal{B} \mid x \in B\}$$

Definition

Springer sheaf defined to be parabolic induction

$$\mathsf{S} := \mu_* \nu^* \mathbb{C}_{\{0\}} = \mu_* \mathbb{C}_{\widetilde{\mathcal{N}}} \in \mathrm{Perv}^G(\mathcal{N})$$

- Historical precursor to character sheaves
- Characters of unipotent reps (inside $R_T^G(1)$) determined on $\mathcal{N}(\mathbb{F}_q)$

(Categorical) Springer theory

Theorem (Borho, MacPherson)

There is an isomorphism of algebras

$$\mathrm{End}_{\mathrm{Perv}^G(\mathcal{N})}(\mathsf{S}) \simeq kW, \qquad \mathsf{S} = \bigoplus_\chi L_\chi \otimes IC_\chi$$

In particular, since $\mathrm{Perv}^G(\mathcal{N})$ *is semisimple, identify Springer block*

$$kW\mathrm{-mod} \simeq \langle \mathsf{S} \rangle \subset \mathrm{Perv}^G(\mathcal{N}).$$

Unipotent principal series $\mathrm{Rep}(G(\mathbb{F}_q), B(\mathbb{F}_q))$ governed by finite Hecke algebra

$$\mathcal{H}_q^{fin} := \mathcal{H}(G(\mathbb{F}_q), B(\mathbb{F}_q)) = \mathrm{End}_{G(\mathbb{F}_q)}(\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} k)$$

Theorem (Tits, Benson, Curtis, Lusztig): $\mathcal{H}_q^{fin} \simeq kW$

Affine Springer theory

- F non-arch. local field (e.g. $\mathbb{F}_q((t)), \mathbb{Q}_p$)
  $O \subset F$ ring of integers (e.g. $\mathbb{F}_q[[t]], \mathbb{Z}_p$)
- Study $\mathrm{Rep}^{sm}(G(F))$ Notation: $G(F) = G_F$
- Iwahori $I = B_{\mathbb{F}_q} \times_{G_{\mathbb{F}_q}} G_O$, e.g. $\begin{pmatrix} O & O & O \\ tO & O & O \\ tO & tO & O \end{pmatrix}$
- Affine character sheaves (Bezrukavnikov, Kazhdan, Varshavsky '18): categorifications of functions/distributions on G_F/G_F, identifies categorical center of $\mathrm{Rep}^{sm}(G_F)$
- Affine Springer theory (Bouthier, Kazhdan, Varshavsky '20): perverse sheaves, intersection cohomology, affine Springer sheaf with $\mathrm{End} \simeq k\widetilde{W}^{aff}$

Langlands duality

- Local Langlands conj.: a surjection ("automorphic" vs. "spectral")
  $\mathrm{Irr}(G_F)\text{``}\leftrightarrow\text{''}\ \mathrm{Hom}_{grp}(WD_F, G^\vee(\mathbb{C}))$ (Langlands parameters)

- Deligne-Langlands

$$\mathrm{Irr}(G_F, I)\text{``}\leftrightarrow\text{''}\{(s,n) \in G^{\vee,ss}(\mathbb{C}) \times G^{\vee,u}(\mathbb{C}) \mid sns^{-1} = n^q\}$$

- Bezrukavnikov's Langlands dual affine Hecke categories

$$D(I\backslash G_F/I) \simeq \mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$$

- Affine Springer theory

$$\mathsf{S} \in \mathrm{Perv}^{G_F}(\mathfrak{g}_F^{nil}) \longleftrightarrow ???$$

Future: Study relationship

Warning

Warning: $G^\vee$ is now G

$k = \mathbb{C}$

Assume: (spectral) G split, simply connected

Derived algebraic geometry

- Replace rings with *differential-graded rings* (ring objects in the category of chain complexes) up to quasi-isomorphism
- Derived fiber product of schemes $X \times_S^R Y$ computed by semi-free resolutions (analogue of free resolution)
- Derived (Artin stacks), etc...
- Example: if $G = T$ is a torus, then *derived Steinberg*:

$$\{0\} \times_{\mathfrak{t}}^R \{0\} = \mathfrak{t}[-1] = \operatorname{Spec} \operatorname{Sym}_k \mathfrak{t}^*[1]$$

  records $\operatorname{Tor}^\bullet_{k[\mathfrak{t}]}(k, k)$

$$\operatorname{Coh}^T(\{0\} \times_{\mathfrak{t}} \{0\}) \simeq \operatorname{Sym}_k \mathfrak{t}^*[1]\text{-mod} \otimes \operatorname{Rep}(T)$$

- Automorphic side:

$$D(T_O^\vee \backslash T_F^\vee / T_O^\vee) = D(T_O^\vee \backslash X_\bullet(T^\vee)) \simeq D(BT_k^\vee) \otimes D(X^\bullet(T))$$

Derived free loop space

Recall: inertia stack $IX(R) = \{x \in X(R), \alpha \in \mathrm{Aut}_{X(R)}(x)\}$.

Definition

The *derived loop space* or *derived inertia stack*:

$$\mathcal{L}X := \mathrm{Map}_{DSt}(S^1, X) = \mathrm{Map}_{DSt}(D^1 \coprod_{S^0} D^1, X) \simeq X \times^R_{X \times X} X.$$

Global quotient stack X/G, "derived universal stabilizer" (or "derived universal fixed points")

$$\begin{array}{ccc} \mathcal{L}(X/G) & \longrightarrow & (X \times G)/G \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & (X \times X)/G \end{array}$$

$$\mathcal{L}(X/G)(\mathbb{C}) = \{(x, g) \in X(\mathbb{C}) \times G(\mathbb{C}) \mid g \cdot x = x\}/G(\mathbb{C})$$

Examples of derived loop spaces

Observation: Stacks on spectral side of Langlands are often derived loop spaces

- X scheme, then $\mathcal{L}X = \mathbb{T}_X[-1] = \mathrm{Spec}_X\, \mathrm{Sym}_X\, \Omega^1_X[1]$ by HKR
- $\mathcal{L}(BG) = \mathcal{L}(*/G) = G/G$
- derived commuting stack: $\mathcal{L}(G/G) \approx \{(g,h) \in G \times G \mid gh = hg\}/G$
- Grothendieck-Springer resolution:
  $\mathcal{L}(BB) = \mathcal{L}(G\backslash G/B) = \{(g, xB) \in G \times G/B \mid gxB = xB\}$
  $= \{(g, B) \in G \times \mathcal{B} \mid g \in B\} = \widetilde{G}/G$
- (group) Steinberg stack: $\mathcal{L}(B\backslash G/B) = \mathcal{L}(\frac{G/B \times G/B}{G}) = \frac{\widetilde{G} \times_G \widetilde{G}}{G}$
- (Deligne-)Langlands parameters:
  $\mathcal{L}(\mathcal{N}/(G \times \mathbb{G}_m)) = \{(g, q, n) \in G \times \mathbb{G}_m \times \mathcal{N} \mid gng^{-1} = qn\}/G$

Coherent Springer sheaf

Notation: For a group K, let $K_+ = K \times \mathbb{G}_m$.

Definition

$\mathcal{L}_q(\mathcal{N}/G_+) := \mathcal{L}(\mathcal{N}/G_+) \times_{\mathbb{G}_m \times B\mathbb{G}_m} \{q\}$

Parabolic induction

$$\mathcal{L}(\mathcal{N}/G_+) \xleftarrow{\mathcal{L}\mu} \mathcal{L}(\widetilde{\mathcal{N}}/G_+) = \mathcal{L}(\mathfrak{n}/B_+) \xrightarrow{\mathcal{L}\nu} \mathcal{L}(\{0\}/H_+)$$

Definition (Coherent Springer sheaf)

$$\mathcal{S} := \mathcal{L}\mu_* \mathcal{L}\nu^* \mathcal{O}_{\mathcal{L}(\{0\}/H_+)} = \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/G_+)} \in D\,\mathrm{Coh}(\mathcal{L}(\mathcal{N}/G_+))$$

$$\mathcal{S}_q := \mathcal{L}_q\mu_* \mathcal{L}_q\nu^* \mathcal{O}_{\mathcal{L}_q(\{0\}/H_+)} = \mathcal{L}_q\mu_* \mathcal{O}_{\mathcal{L}_q(\widetilde{\mathcal{N}}/G_+)} \in D\,\mathrm{Coh}(\mathcal{L}_q(\mathcal{N}/G_+))$$

Coherent Springer theory

Iwahori-Hecke algebra: $\mathcal{H}_q^{aff} = \mathcal{H}(G_F, I)$ governs $\mathrm{Rep}(G_F, I)$

Affine Hecke algebra: assemble into an algebraic family $\mathcal{H}^{aff}$ in q

Theorem (Ben-Zvi, -, Helm, Nadler)

Assume $q \neq 1$. There are isomorphisms of dg algebras

$$R\,\mathrm{End}_{\mathcal{L}(\mathcal{N}/G_+)}(\mathcal{S}) \simeq \mathcal{H}^{aff}, \qquad R\,\mathrm{End}_{\mathcal{L}_q(\mathcal{N}/G_+)}(\mathcal{S}_q) \simeq \mathcal{H}_q^{aff}$$

In particular,

$$D_{perf}(\mathcal{H}^{aff}\text{-mod}) \simeq \langle \mathcal{S} \rangle \subset D\,\mathrm{Coh}(\mathcal{L}(\mathcal{N}/G_+)),$$
$$D_{perf}(\mathcal{H}_q^{aff}\text{-mod}) \simeq \langle \mathcal{S}_q \rangle \subset D\,\mathrm{Coh}(\mathcal{L}_q(\mathcal{N}/G_+)).$$

When $q = 1$, get $kW^{aff} \otimes_k \mathrm{Sym}_k(\mathfrak{h}[-1] \oplus \mathfrak{h}[-2])$

Current progress

- Eugen Hellmann: On the derived category of the Iwahori-Hecke algebra (Conjecture 3.2)
- Tamir Hemo and Xinwen Zhu: forthcoming work

Outline of argument

Definition

derived Steinberg $\mathcal{Z} := \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}} = \{(x, B, B') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in B' \cap B\}$
affine Hecke category $\mathsf{H}^{aff} := \mathsf{Coh}(\mathcal{Z}/G)$
mixed affine Hecke category $\mathsf{H}^{aff}_m := \mathsf{Coh}(\mathcal{Z}/G_+)$

Deligne-Langlands (KL '87)	new features
$K_0(\mathsf{H}^{aff}_m) \simeq \mathcal{H}^{aff}$	$HH(\mathsf{H}^{aff}_m) \simeq \mathcal{H}^{aff}$ $\simeq \omega(\mathcal{L}(\mathcal{Z}/G_+))$
localization $K_0(\mathsf{H}^{aff}_m)_a \simeq H^{BM}_\bullet(\mathcal{Z}^a)$	localization of loop spaces $\mathcal{L}_a(\mathcal{Z}/G_+) \simeq \mathcal{L}(\mathcal{Z}^a_\circ)$
sheaf theory $H^{BM}_\bullet(\mathcal{Z}^a) \simeq \mathrm{Ext}^\bullet(\mu^a_* \mathbb{C}_{\widetilde{\mathcal{N}}^a})$	$\omega(\mathcal{L}(\mathcal{Z}/G_+)) \simeq \mathrm{Ext}^\bullet(\mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/G_+)})$ $\mathcal{D} - \Omega$ Koszul duality

Hochschild homology as a trace

U symmetric monoidal ∞-category ($\mathsf{U} = \mathsf{dgCat}^{cont}$)

Definition

$X \in \mathsf{U}$ *dualizable* if there is $X^\vee$ with coevaluation $\eta : 1 \to X^\vee \otimes X$, evaluation $\epsilon : X^\vee \otimes X \to 1$ satisfying identities

Definition

For $f : X \to X$, the *trace* is

$$\mathrm{tr}(X, f) = \epsilon \circ (f \otimes 1) \circ \eta \in \mathrm{End}_{\mathsf{U}}(1), \qquad \dim(X) = \mathrm{tr}(X, \mathrm{id}_X)$$

- Dual is unique up to unique isomorphism, so trace is independent of choices.
- Toy example: $\mathsf{U} = \mathsf{Vect}$ recovers usual trace of matrices

Trace of dg categories

$\mathsf{U} = \mathsf{dgCat}^{cont}$

- cocomplete dg categories, functors commuting with colimits, Lurie $\otimes$
- $1 = \mathsf{Vect}$ $\quad \mathsf{End}^{cont}_{\mathsf{dgCat}}(\mathsf{Vect}) = \mathsf{Vect}$
- Every compactly generated dg category is dualizable (Yoneda pairing)

Definition

The *Hochschild homology* of a dg category C, $F : \mathsf{C} \to \mathsf{C}$

$$HH(\mathsf{C}, F) := \mathrm{tr}(\mathsf{C}, F) \qquad HH(\mathsf{C}) = \mathrm{tr}(\mathsf{C}, \mathrm{id}).$$

Functorial for (continuous) functors preserving compact objects

Consequence: A, F monoidal, $1_{\mathsf{A}} \in \mathsf{A}$ compact, $\circ : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A}$ preserves compactness

$$\Rightarrow HH(\mathsf{A}, F) \text{ is an algebra}$$

Algebraic model: dg Morita theory

$$HH(A\text{-mod}) = HH(A) = A \otimes^L_{A \otimes A^{op}} A$$

Categorical invariant:

- If X is a compact generator of C, then $HH(\mathsf{C}) = HH(\mathrm{End}^\bullet_{\mathsf{C}}(X))$
- Map from K-theory spectrum $K_\bullet(\mathsf{C}) \to HH_\bullet(\mathsf{C})$
  - Factors through $K_0(\mathsf{C})$ if $HH_\bullet(\mathsf{C})$ coconnective
- HH takes semiorthogonal decompositions to $\bigoplus$

Theorem (Ben-Zvi, -, Helm, Nadler)

For $q \neq 1$, we have equivalences:

$$\mathcal{H}^{aff} \simeq K_0(\mathsf{H}^{aff}_m) \to HH(\mathsf{H}^{aff}_m)$$
$$\mathcal{H}^{aff}_q \simeq K_0(\mathsf{H}^{aff}_m) \otimes_{k[z,z^{-1}]} k_q \to HH(\mathsf{H}^{aff}, q^*).$$

Sketch of argument

Theorem (Bezrukavnikov)

$F = \overline{\mathbb{F}_q}((t))$, $I \subset G^\vee_F$ Iwahori subgroup

$$\mathrm{H}^{aff} = \mathrm{Coh}(\mathcal{Z}/G) \simeq D(I\backslash G^\vee_F/I)$$

intertwining q^* with Fr^*

- Stratification by I-orbits on $G^\vee(F)/I \rightsquigarrow$ semiorthog. decomp. of H^{aff}
- Lift to a semi-orthogonal decomposition of H^{aff}_m
- HH coconnective on each summand
- $K_0 \to HH$ is isomorphism on each summand

Geometric model: functions on loop spaces

Let X be a perfect stack over k

$\mathsf{C} = \mathrm{QCoh}(X) \qquad \mathsf{C}^\vee = \mathrm{QCoh}(X)$

Theorem (Ben-Zvi, Francis, Nadler)

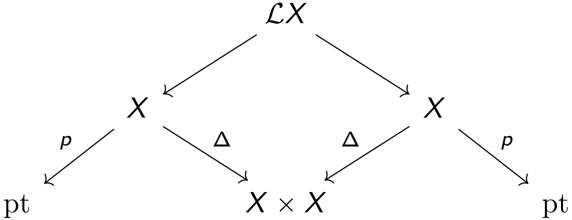
Let X be a perfect stack. There is a monoidal equivalence

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \simeq \mathrm{Fun}_k^L(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

by integral kernels: $F_{\mathcal{K}}(-) = Rp_{2*}(\mathcal{K} \otimes^L p_1^*(-))$

Similar for IndCoh, !-transforms

Geometric model: functions on loop spaces



Coevaluation: $\Delta_* p^* k = \Delta_* \mathcal{O}_X$ Evaluation: $p_* \Delta^*(-)$

Derived base change:

$$HH(\mathrm{QCoh}(X)) = \mathcal{O}(\mathcal{L}X)$$

$$HH(\mathrm{IndCoh}(X)) = \omega(\mathcal{L}X)$$

Example: $\mathrm{QCoh}(\mathbb{P}^1)$

Probe HH_0 using map from K_0

Geometry: Use HKR

$$K_0(\mathrm{Coh}(\mathbb{P}^1)) \to HH_0(\mathrm{QCoh}(\mathbb{P}^1)) = H^0(\mathbb{P}^1, \mathcal{O}) \bigoplus H^1(\mathbb{P}^1, \Omega^1) = k^2$$

$$k \to \Gamma(X, \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})) \simeq \Gamma(X, \Delta^*(\mathcal{E}^\vee \boxtimes \mathcal{E})) \to \Gamma(X, \Delta^*\Delta_*\mathcal{O}_X) = \mathcal{O}(\mathcal{L}X)$$

$$[\mathcal{O}(n)] \mapsto (1, -n)$$

Algebra: Choose generators $\mathcal{O}(-1), \mathcal{O}$

$$K_0(\mathrm{Coh}(\mathbb{P}^1)) \to \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(-1)) \bigoplus \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = k^2$$

$$\mathcal{O}(-1) \mapsto (1, 0), \qquad \mathcal{O} \mapsto (0, 1)$$

$$[\mathcal{O}(n)] \mapsto (\chi(\mathcal{L}(1)), \chi(\mathcal{L})) = (-n, n+1)$$

Convolution categories

Set-up: X, Y smooth geometric stacks, and $f : X \to Y$ proper. Let $Z = X \times_Y X$.

Example: $\mathsf{H}_m^{aff} = \mathsf{Coh}^{G_+}(\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}})$

What is convolution algebra structure on $\omega(\mathcal{L}(X \times_Y X))$?

$$\mathrm{Hom}_{\mathcal{L}(X \times_Y X)}(\mathcal{O}_{\mathcal{L}(X \times_Y X)}, \omega_{\mathcal{L}(X \times_Y X)}) \simeq \mathrm{Hom}_{\mathcal{L}X}(\mathcal{O}_{\mathcal{L}X}, p_* p^! \omega_{\mathcal{L}X})$$

$$\simeq \mathrm{Hom}_{\mathcal{L}X}(\mathcal{O}_{\mathcal{L}X}, f^! f_* \omega_{\mathcal{L}X}) \simeq \mathrm{Hom}_{\mathcal{L}Y}(f_* \mathcal{O}_{\mathcal{L}X}, f_* \omega_{\mathcal{L}X}) \simeq \mathrm{End}_{\mathcal{L}Y}(f_* \mathcal{O}_{\mathcal{L}X})$$

Since X smooth, $\mathcal{O}_{\mathcal{L}X} \simeq \omega_{\mathcal{L}X}$ (derived Calabi-Yau)

Hochschild homology of convolution categories

Proposition

We have an equivalence of algebras

$$HH(\mathrm{Coh}(X \times_Y X)) \simeq \mathrm{End}_{\mathcal{L}Y}(\mathcal{L}f_* \mathcal{O}_{\mathcal{L}X}).$$

Uses theory of higher traces by Gaitsgory, Kazhdan, Rozenblyum, Varshavsky

Lemma

The category $\mathrm{IndCoh}(X \times_Y X)$ *is monoidal under convolution, and it is rigid (compact objects have compact left and right duals).*

Coherent Springer theory

Theorem (Ben-Zvi, -, Helm, Nadler)

There is an isomorphism of dg algebras

$$R\,\mathrm{End}_{\mathcal{L}(\mathcal{N}/G_+)}(\mathcal{S}) \simeq HH(\mathrm{H}_m^{aff}) \simeq K_0(\mathrm{H}_m^{aff}) \simeq \mathcal{H}^{aff}.$$

In particular,

$$D(\mathcal{H}^{aff}\text{-mod}) \simeq \langle \mathcal{S} \rangle \subset \mathrm{IndCoh}(\mathcal{L}(\mathcal{N}/G_+))$$

Theorem (Ben-Zvi, -, Helm, Nadler)

For $q \neq 1$,

$$R\,\mathrm{End}_{\mathcal{L}_q(\mathcal{N}/G_+)}(\mathcal{S}_q) \simeq HH(\mathrm{H}^{aff}, q^*) \simeq \mathcal{H}_q^{aff}$$

$$D(\mathrm{Rep}^{sm}(G_F, I)) \simeq D(\mathcal{H}_q^{aff}\text{-mod}) \simeq \langle \mathcal{S}_q \rangle \subset \mathrm{IndCoh}(\mathcal{L}_q(\mathcal{N}/G_+)).$$

Case $G = GL_n$

Definition

Stack of Langlands parameters X_{F,GL_n}: pairs $\rho : W_F \to GL_n(\mathbb{C})$ and $N \in \mathcal{N} \subset \mathfrak{g}$ such that $\mathrm{Ad}_{\rho(\mathsf{Fr}^n \sigma)}(N) = q^n N$ for $\sigma \in I_F$

$X_{F,GL_n} = \coprod_\nu X^\nu_{F,GL_n}$ where ν is a "type" (i.e. $\rho|_{I_F}$ up to conjugation)

Example: $X^1_{F,GL_n} = \mathcal{L}_q(\mathcal{N}/GL_{n,+})$

Theorem (Ben-Zvi, -, Helm, Nadler)

There is an embedding

$$D(\mathsf{Rep}^{sm}(GL_{n,F})) \hookrightarrow \mathsf{IndCoh}(X_{F,GL_n}).$$

Conjecture: it is an equivalence

Equivariant localization

Deligne-Langlands (Kazhdan, Lusztig): classify $\mathrm{Irr}(\mathcal{H}^{aff})$

$$K_0(\mathcal{Z}/G_+) \otimes_{K_0(BG_+)} \mathbb{C}_a \simeq H_\bullet^{BM}(\mathcal{Z}^a)$$

Observation: $\mathcal{L}(X/G) \to G/G$, classical fiber over $g \in G$ is X^g

Theorem (Equivariant localization for loop spaces)

$z \in G$ semisimple in reductive, and X as below. There is an equivalence

$$\widehat{\mathcal{L}}_z(X/G) \simeq \widehat{\mathcal{L}}(X^z_\circ/G^z).$$

Informally, near semisimple $z \in \mathcal{L}(BG)$, the loop space $\mathcal{L}(X/G)$ looks like the loop space of the z-fixed points.

- If X is smooth, then $X^z_\circ = \pi_0(X^z)$ (smooth)
- If $X = Y \times_S W$ for Y, S, W smooth, then $X^z_\circ = \pi_0(Y^z) \times_{\pi_0(S^z)} \pi_0(W^z)$ (derived lci or quasismooth)

Koszul duality for loop spaces

Localization for loop spaces $\rightsquigarrow$ "sheafy" localization formula for HH

Connection to perverse sheaves by $\mathcal{D} - \Omega$ Koszul duality

Observation: $\mathcal{O}_{\mathcal{L}X} \simeq \mathrm{Sym}_X \Omega^1_X[1]$, d_{dR} encoded by S^1-action

Theorem (Koszul duality for loop spaces)

Assume X is smooth. There is an equivalence

$$\mathrm{KPerf}(\widehat{\mathcal{L}}(X/G))^{S^1 \rtimes \mathbb{G}_m} \simeq F\mathcal{D}_{coh}(X/G)$$

between a certain S^1-equivariant small subcategory of $\mathrm{IndCoh}(\widehat{\mathcal{L}}(X/G))$ *and the (derived) category of coherent filtered D-modules on X/G.*

Kapranov, Beilinson, Drinfeld, Ben-Zvi, Nadler, Positselski: derived filtered D-modules equivalent to coderived modules for dg de Rham algebra

Specialization at parameters

Corollary

Let $a \in G_+$ be semisimple, and $\mathsf{S}(a)$ denote the (weakly equivariant) a-fixed Springer sheaf with derived endomorphisms $\mathcal{H}^{aff}(a)$. There is a map of dg algebras $\mathcal{H}^{aff} \to \mathcal{H}^{aff}(a)$, and a commuting diagram

$$\begin{array}{ccc} D(\mathcal{H}^{aff}\text{-mod}) \simeq \langle S \rangle & \hookrightarrow & \mathrm{IndCoh}(\mathcal{L}(\mathcal{N}/G_+)) \\ \uparrow & & \uparrow \\ D(\mathcal{H}^{aff}(a)\text{-mod}) \simeq \langle \mathsf{S}(a) \rangle & \hookrightarrow & F\mathcal{D}^w(\mathcal{N}^a/G_+^a) \end{array}$$

Future: use this to study t-structures, simple objects, etc.

Thanks

Thanks!