Finite Howe Correspondence and Lusztig Classification

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Abstract

Let (G, G') be a reductive dual pair inside a finite symplectic group. By restricting the Weil representation to the dual pair, there exists a relation (called the finite Howe correspondence) between the irreducible representations of the two groups G, G'. In this talk, we would like to discuss some progress on the understanding of the correspondence by using Lusztig's classification on the representations of finite classical groups. In particular, we will focus on the following three subjects:

- the decomposition of the uniform projection of the Weil character
- the "commutativity" between the Howe correspondence and the Lusztig correspondence
- the description of the Howe correspondence on unipotent characters in terms of the symbols by Lusztig.

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Weil characters

- $\mathbf{f}_q$: a finite field of q elements, q odd
- $\mathbf{G}$: a classical group over $\mathbf{f}_q$, e.g., GL_n, U_n, Sp_{2n}, O_n^ϵ
- $F\colon \mathbf{G} \to \mathbf{G}$, the *Frobenius endomorphism*, $G = \mathbf{G}^F$ a finite classical group
- $\mathcal{V}(G)$: the (complex) space of class functions on G with inner product $\langle , \rangle$
- $\mathcal{E}(G)$: the set of irreducible characters of G, an orthonormal basis for $\mathcal{V}(G)$
- $\omega^\psi = \omega^\psi_{\mathrm{Sp}_{2N}}$, (the character of) *Weil representation* of $\mathrm{Sp}_{2N}(q)$ where ψ a nontrivial (additive) character of $\mathbf{f}_q$

Dual pair and Howe correspondence

- $(\mathbf{G}, \mathbf{G}')$ a *reductive dual pair*, e.g. $(\mathrm{GL}_n, \mathrm{GL}_{n'})$, $(\mathrm{U}_n, \mathrm{U}_{n'})$, $(\mathrm{Sp}_{2n}, \mathrm{O}^{\epsilon}_{n'})$
- $\omega^{\psi}_{\mathbf{G},\mathbf{G}'}$: *Weil character* of $G \times G'$ by the homomorphism $G \times G' \to GG' \hookrightarrow \mathrm{Sp}_{2nn'}(q)$
- $\omega^{\psi}_{\mathbf{G},\mathbf{G}'} \in \mathcal{V}(G \times G')$, then

$$\omega^{\psi}_{\mathbf{G},\mathbf{G}'} = \sum_{\rho \in \mathcal{E}(G), \rho' \in \mathcal{E}(G')} m_{\rho,\rho'} \rho \otimes \rho'$$

  where $m_{\rho,\rho'} \in \mathbb{N} \cup \{0\}$ (usually $m_{\rho,\rho'} = 0, 1$)
- the *Howe correspondence* for (G, G'):

$$\Theta_{\mathbf{G},\mathbf{G}'} = \{ (\rho, \rho') \in \mathcal{E}(G) \times \mathcal{E}(G') \mid m_{\rho,\rho'} \neq 0 \}$$
$$\Theta_{\mathbf{G}'}(\rho) = \{ \rho' \in \mathcal{E}(G') \mid (\rho, \rho') \in \Theta_{\mathbf{G},\mathbf{G}'} \}$$

- to understand $\Theta_{\mathbf{G},\mathbf{G}'}$ means that to find the coordinates of $\omega^{\psi}_{\mathbf{G},\mathbf{G}'}$ with respect to the orthonormal basis $\mathcal{E}(G \times G')$

Deligne-Lusztig virtual characters and Uniform projection

- $R^{\mathbf{G}}_{\mathbf{T},\theta} = R^{\mathbf{G}}_{\mathbf{T}^*,s}$ the *Deligne-Lusztig virtual character* of G, $\theta \in \mathcal{E}(T)$, $s \in T^*$
- $R^{\mathrm{O}^\epsilon_n}_{\mathbf{T}^*,s} = \mathrm{Ind}^{\mathrm{O}^\epsilon_n}_{\mathrm{SO}^\epsilon_n}(R^{\mathrm{SO}^\epsilon_n}_{\mathbf{T}^*,s})$
- $\mathcal{V}(G)^\sharp$ the subspace spanned by all $R^{\mathbf{G}}_{\mathbf{T}^*,s}$
- $f^\sharp$ the *uniform projection* of $f \in \mathcal{V}(G)$, f is *uniform* if $f = f^\sharp$.
- $\mathcal{V}(G)^\sharp = \mathcal{V}(G)$ if $\mathbf{G} = \mathrm{GL}_n, \mathrm{U}_n$
- $\mathcal{V}(G)^\sharp \neq \mathcal{V}(G)$ if $\mathbf{G} = \mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_n$
- 1_G is uniform if G connected, but

$$1^\sharp_{\mathrm{O}^\epsilon} = \mathrm{sgn}^\sharp_{\mathrm{O}^\epsilon} = \frac{1}{2}(1_{\mathrm{O}^\epsilon} + \mathrm{sgn}_{\mathrm{O}^\epsilon}),$$

not uniform

Decomposition of the Weil character I

- If $n' > n$, then

$$\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'}} = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{|W_k|}\frac{1}{|\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}|}\frac{1}{|\mathbf{W}_{\mathrm{SO}_{2(n'-k)}}|}\sum_{v\in W_k}\sum_{s\in T_v^*}\sum_{w\in\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}}\sum_{w'\in\mathbf{W}_{\mathrm{SO}^{\epsilon_v\epsilon}_{2(n'-k)}}}\epsilon_v R^{\mathrm{Sp}_{2n}}_{\mathbf{T}_v^*\times\mathbf{T}_w^*,(s,1)}\otimes R^{\mathrm{O}^\epsilon_{2n'}}_{\mathbf{T}_v^*\times\mathbf{T}_{w'}^*,(s,1)};$$

- if $n' \le n$, then

$$\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'}} = \frac{1}{2}\sum_{k=0}^{n'-1}\frac{1}{|W_k|}\frac{1}{|\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}|}\frac{1}{|\mathbf{W}_{\mathrm{SO}_{2(n'-k)}}|}\sum_{v\in W_k}\sum_{s\in T_v^*}\sum_{w\in\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}}\sum_{w'\in\mathbf{W}_{\mathrm{SO}^{\epsilon_v\epsilon}_{2(n'-k)}}}\epsilon_v R^{\mathrm{Sp}_{2n}}_{\mathbf{T}_v^*\times\mathbf{T}_w^*,(s,1)}\otimes R^{\mathrm{O}^\epsilon_{2n'}}_{\mathbf{T}_v^*\times\mathbf{T}_{w'}^*,(s,1)}$$
$$+\frac{1}{2}\frac{1}{|W^\epsilon_{n'}|}\frac{1}{|\mathbf{W}_{\mathrm{Sp}_{2(n-n')}}|}\sum_{v\in W^\epsilon_{n'}}\sum_{s\in T_v^*}\sum_{w\in\mathbf{W}_{\mathrm{Sp}_{2(n-n')}}}\epsilon R^{\mathrm{Sp}_{2n}}_{\mathbf{T}_v^*\times\mathbf{T}_w^*,(s,1)}\otimes R^{\mathrm{O}^\epsilon_{2n'}}_{\mathbf{T}_v^*,s}$$

Decomposition of the Weil character II

- $$\begin{aligned}&\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1}}\cdot(1\otimes\chi_{\mathrm{O}_{2n'+1}})\\&=\frac{1}{2}\sum_{k=0}^{\min(n,n')}\frac{1}{|W_k|}\frac{1}{|\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}|}\frac{1}{|\mathbf{W}_{\mathrm{SO}_{2(n'-k)+1}}|}\sum_{v\in W_k}\sum_{s\in T_v^*}\\&\sum_{w\in\mathbf{W}_{\mathrm{Sp}_{2(n-k)}}}\sum_{w'\in\mathbf{W}_{\mathrm{SO}_{2(n'-k)+1}}}\epsilon_w R^{\mathrm{Sp}_{2n}}_{\mathbf{T}^*_v\times\mathbf{T}^*_w,(s,-1)}\otimes R^{\mathrm{O}^\epsilon_{2n'+1}}_{\mathbf{T}^*_v\times\mathbf{T}^*_{w'},(s,-1)}\end{aligned}$$

  where $\chi_{\mathrm{O}_{2n'+1}}$ is the linear character determined by the spinor norm
- $\mathbf{T}_w$, the maximal torus corresponding to $w\in W_{\mathbf{G}}$, $W_n=W_n^+\sqcup W_n^-$, $\epsilon_v=\pm1$ depending on $v\in W_n^\pm$
- $(\mathrm{GL}_n,\mathrm{GL}_{n'})$, $(\mathrm{U}_n,\mathrm{U}_{n'})$, $(\mathrm{Sp}_{2n'},\mathrm{O}^\epsilon_{2n'})$ and $q>>0$ (Srinivasan 1979)
- $(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1})$ and $q>>0$ (P. 2016)
- without $q>>0$ (P. 2021)

Lusztig Series

- Lusztig shows that

$$\mathcal{E}(G)=\bigsqcup_{(s)\subset(G^*)^0}\mathcal{E}(G)_s,\qquad \mathcal{V}(G)^\sharp=\bigoplus_{(s)\subset(G^*)^0}\mathcal{V}(G)^\sharp_s$$

- $\mathcal{E}(G)_s=\{\,\rho\in\mathcal{E}(G)\mid\langle\rho,R^{\mathbf{G}}_{\mathbf{T}^*,s}\rangle_G\neq 0$ for some $T^*\ni s\,\}$, a *Lusztig series*,
- elements in $\mathcal{E}(G)_1$ are *unipotent*
- $\mathcal{V}(G)_s$: the subspace spanned by $\mathcal{E}(G)_s$.
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$$\omega^\psi_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'}}=\sum_{l=0}^{\min(n,n')}\sum_{(s)\subset X_{l,l}}\omega^\psi_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},s}.$$

where $\omega^\psi_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},s}$ denotes the orthogonal projection of $\omega^\psi_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'}}$ over $\mathcal{V}(\mathrm{Sp}_{2n})_{(s,1)}\otimes\mathcal{V}(\mathrm{O}^\epsilon_{2n'})_{(s,1)}$

Lusztig Correspondence

- Lusztig shows that there exists a bijection

$$\mathfrak{L}_s\colon \mathcal{E}(G)_s \to \mathcal{E}(C_{G^*}(s))_1$$

such that

$$\langle \eta, \epsilon_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}^*,s}\rangle_G = \langle \mathfrak{L}_s(\eta), \epsilon_{C_{\mathbf{G}^*}(s)} R^{C_{\mathbf{G}^*}(s)}_{\mathbf{T}^*,\mathbf{1}}\rangle_{C_{G^*}(s)}$$

- $\mathfrak{L}_s$ called a *Lusztig correspondence*, can be extended to a vector space isomorphism $\mathfrak{L}_s\colon \mathcal{V}(G)_s \to \mathcal{V}(C_{G^*}(s))_1$
- $\mathfrak{L}_s$ is not uniquely determined, but $\mathfrak{L}_s(\rho^\sharp) = \mathfrak{L}_s(\rho)^\sharp$
- if $s = 1$, then $\mathcal{E}(G)_1 \simeq \mathcal{E}(G^*)_1$

Centralizer

- for s, we can define $\mathbf{G}^{(0)}$, $\mathbf{G}^{(-)}$, $\mathbf{G}^{(+)}$ so that there is a (modified) Lusztig correspondence

$$\Xi_s\colon \mathcal{E}(G)_s \to \begin{cases} \mathcal{E}(G^{(0)}\times G^{(-)}\times G^{(+)})_1\times\{\pm 1\}, & \text{if } G \text{ is odd-orthogonal;}\\ \mathcal{E}(G^{(0)}\times G^{(-)}\times G^{(+)})_1, & \text{otherwise}\end{cases}$$

- $\mathbf{G}^{(0)}$ is a product of groups of types GL or U;
- $$(\mathbf{G}^{(-)},\mathbf{G}^{(+)}) = \begin{cases} (\mathrm{O}^{\epsilon'}_{2n^{(-)}},\mathrm{O}^{\epsilon''}_{2n^{(+)}}), & \text{if } \mathbf{G}=\mathrm{O}^{\epsilon}_{2n};\\ (\mathrm{O}^{\epsilon'}_{2n^{(-)}},\mathrm{Sp}_{2n^{(+)}}), & \text{if } \mathbf{G}=\mathrm{Sp}_{2n};\\ (\mathrm{Sp}_{2n^{(-)}},\mathrm{Sp}_{2n^{(+)}}), & \text{if } \mathbf{G}=\mathrm{O}^{\epsilon}_{2n+1}.\end{cases}$$
- $\Xi_s(\rho)=\rho^{(0)}\otimes\rho^{(-)}\otimes\rho^{(+)}$ or $\rho^{(0)}\otimes\rho^{(-)}\otimes\rho^{(+)}\otimes\varepsilon$

Weil character under Lusztig correspondence

$$\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'}} = \sum_{l=0}^{\min(n,n')} \sum_{(s)\subset X_{l,l}} \omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},s}$$

$$\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1}} \cdot (1\otimes\chi_{\mathrm{O}^\epsilon_{2n'+1}}) = \sum_{l=0}^{\min(n,n')} \sum_{(s)\subset X_{l,l}} (\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1},s,+} + \omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1},s,-})$$

$$\Xi_{((s,1),(s,1))}(\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},s}) = R^\sharp_{\mathbf{G}^{(0)},1}\otimes R^\sharp_{\mathbf{G}^{(-)},1}\otimes \omega^\sharp_{\mathbf{G}^{(+)},\mathbf{G}'^{(+)},1}$$

$$\Xi_{((s,-1),(s,-1))}(\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1},s,\varepsilon}) = R^\sharp_{\mathbf{G}^{(0)},1}\otimes \omega^\sharp_{\mathbf{G}^{(-)},\mathbf{G}'^{(-)},1}\otimes R^\sharp_{\mathbf{G}^{(+)},1}$$

where $R_{\mathbf{G},1} = \sum_{\rho\in\mathcal{E}(G)_1}\rho\otimes\rho$ and $R^\sharp_{\mathbf{G},1} = \frac{1}{|\mathbf{W}_\mathbf{G}|}\sum_{w\in\mathbf{W}_\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}^*_w,1}\otimes R^{\mathbf{G}}_{\mathbf{T}^*_w,1}$.

Howe Correspondence and Lusztig Correspondence

Theorem (P.)

Let $(\mathbf{G},\mathbf{G}')$ *be a finite reductive dual pair. Suppose* $\rho\in\mathcal{E}(G)_s$, $\rho'\in\mathcal{E}(G')_{s'}$, *and* $(\rho,\rho')\in\Theta_{\mathbf{G},\mathbf{G}'}$.

1. *Suppose that* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},)$ *and write* $\Xi_s(\rho)=\rho^{(0)}\otimes\rho^{(-)}\otimes\rho^{(+)}$, $\Xi_{s'}(\rho')=\rho'^{(0)}\otimes\rho'^{(-)}\otimes\rho'^{(+)}$. *Then*
   - $\mathbf{G}^{(0)}\simeq\mathbf{G}'^{(0)}$ *and* $\rho^{(0)}\simeq\rho'^{(0)}$,
   - $\mathbf{G}^{(-)}\simeq\mathbf{G}'^{(-)}$ *and* $\rho^{(-)}\simeq\rho'^{(-)}$,
   - $(\mathbf{G}^{(+)},\mathbf{G}'^{(+)})$ *forms a dual pair and* $(\rho^{(+)}),\rho'^{(+)})\in\Theta_{\mathbf{G}^{(+)},\mathbf{G}'^{(+)}}$.
2. *Suppose that* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'+1})$ *and write* $\Xi_s(\rho)=\rho^{(0)}\otimes\rho^{(-)}\otimes\rho^{(+)}$, $\Xi_{-s'}(\rho'\cdot\chi_{\mathrm{O}^\epsilon_{2n'+1}})=\rho'^{(0)}\otimes\rho'^{(-)}\otimes\rho'^{(+)}$. *Then*
   - $\mathbf{G}^{(0)}\simeq\mathbf{G}'^{(0)}$ *and* $\rho^{(0)}\simeq\rho'^{(0)}$,
   - $(\mathbf{G}^{(-)},\mathbf{G}'^{(-)})$ *forms a dual pair and* $(\rho^{(-)}),\rho'^{(-)})\in\Theta_{\mathbf{G}^{(-)},\mathbf{G}'^{(-)}}$,
   - $\mathbf{G}^{(+)}\simeq\mathbf{G}'^{(+)}$ *and* $\rho^{(+)}\simeq\rho'^{(+)}$.

Example

- $\mathrm{Sp}_{2n}(q)$, $n \geq 1$, has two irreducible characters χ_1, χ_2 of dimension $\frac{q^n+1}{2}$
- both are in $\mathcal{E}(G)_s$ such that $\mathbf{G}^{(1)} = \mathrm{U}_0$, $\mathbf{G}^{(-)} = \mathrm{O}^+_{2n}$ and $\mathbf{G}^{(+)} = \mathrm{Sp}_0$, and

$$\Xi_s\colon \{\chi_1, \chi_2\} \mapsto \{\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathrm{sgn} \otimes \mathbf{1}\}$$

- if $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_{2n'})$, the correspondences of χ_1, χ_2 can be reduced to the correspondence of $\mathbf{1}$ for $(\mathrm{Sp}_0, \mathrm{O}^\epsilon_{2(n'-n)})$, in particular, both χ_1, χ_2 occur only if $n' \geq n$
- if $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_{2n'+1})$, the correspondences of χ_1, χ_2 can be reduced to the correspondences of $\mathbf{1} = \rho_{\binom{n}{0}}, \mathrm{sgn} = \rho_{\binom{0}{n}}$ for $(\mathrm{O}^+_{2n}, \mathrm{Sp}_{2n'})$, in particular, one of χ_1, χ_2 occurs if $n' \geq 0$, and the other occurs only if $n' \geq n$

Some Remarks

- Howe correspondence is compatible with the Lusztig correspondence, and so general Howe correspondence can be reduced to the correspondence on unipotent characters
- for $(\mathrm{GL}_n, \mathrm{GL}_{n'})$ or $(\mathrm{U}_n, \mathrm{U}_{n'})$, (Aubert-Michel-Rouquier 1996)
- for $(\mathrm{Sp}_{2n}, \mathrm{O}^{\epsilon}_{2n'})$ or $(\mathrm{Sp}_{2n}, \mathrm{O}^{\epsilon}_{2n'+1})$, we need to recover $(\rho, \rho') \in \Theta_{\mathbf{G},\mathbf{G}'}$ from $\rho^\sharp \otimes \rho'^\sharp$ inside $\omega^\sharp_{\mathbf{G},\mathbf{G}'}$

Symbols

- a (reduced) *symbol* is an ordered pair
$$\Lambda = \binom{S}{T} = \binom{s_1, s_2, \ldots, s_k}{t_1, t_2, \ldots, t_l}$$
where S, T are finite subsets of $\mathbb{N} \cup \{0\}$, $0 \notin S \cap T$, $\langle s_i \rangle, \langle t_j \rangle$ strictly decreasing, *transpose*: $\Lambda^{\mathrm{t}} = \binom{T}{S}$
- *rank* of Λ is
$$\mathrm{rk}(\Lambda) = \sum_{s \in S} s + \sum_{t \in T} t - \left\lfloor \left(\frac{|S| + |T| - 1}{2} \right)^2 \right\rfloor$$
- *defect* of Λ is $\mathrm{def}(\Lambda) = |S| - |T|$
- let $\mathcal{S}_{n,\delta} = \{\, \Lambda \mid \mathrm{rk}(\Lambda) = n,\ \mathrm{def}(\Lambda) = \delta \}$

Parametrization of Unipotent Characters by Lusztig

- define

$$\mathcal{S}_{\mathrm{Sp_{2n}}} = \bigcup_{\delta\equiv 1 \pmod 4} \mathcal{S}_{n,\delta};$$
$$\mathcal{S}_{\mathrm{O}^+_{2n}} = \bigcup_{\delta\equiv 0 \pmod 4} \mathcal{S}_{n,\delta};$$
$$\mathcal{S}_{\mathrm{O}^-_{2n}} = \bigcup_{\delta\equiv 2 \pmod 4} \mathcal{S}_{n,\delta}$$

- Lusztig shows that there exists a one-to-one correspondence $\mathcal{S}_{\mathbf{G}} \to \mathcal{E}(G)_1$, $\Lambda \mapsto \rho_\Lambda$

Symbols and bi-partitions

- *partition*: $\lambda = [\lambda_i] = [\lambda_1, \lambda_2, \ldots, \lambda_k]$, $\lambda_i \geq 0$, decreasing, $\mathcal{P}(n)$ the set of all partitions n
- *bi-partition* $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ where λ, μ are partitions
- $\mathcal{P}_2(n) = \{ \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \mid |\lambda| + |\mu| = n \}$ the set of bi-partitions of n
- $$\Upsilon\colon \begin{pmatrix} s_1, s_2, \ldots, s_k \\ t_1, t_2, \ldots, t_l \end{pmatrix} \mapsto \begin{bmatrix} s_1 - (k-1), s_2 - (k-2), \ldots, s_k \\ t_1 - (l-1), t_2 - (l-2), \ldots, t_l \end{bmatrix}$$

  gives a bijection

  $$\mathcal{S}_{n,\delta} \to \begin{cases} \mathcal{P}_2(n - (\frac{\delta+1}{2})(\frac{\delta-1}{2})) & \text{if } \delta \text{ odd;} \\ \mathcal{P}_2(n - (\frac{\delta}{2})^2) & \text{if } \delta \text{ even,} \end{cases}$$

  in particular, bijections $\mathcal{S}_{n,1} \to \mathcal{P}_2(n)$ and $\mathcal{S}_{n,0} \to \mathcal{P}_2(n)$

$$\mathcal{E}(\mathrm{Sp}_{2n})_1 \simeq \mathcal{S}_{\mathrm{Sp}_{2n}} \simeq \mathcal{P}_{\mathrm{Sp}_{2n}} = \bigsqcup_{0\leq t,\ t(t+1)\leq n} \mathcal{P}_2(n-t(t+1))$$

$$\mathcal{E}(\mathrm{O}^+_{2n})_1 \simeq \mathcal{S}_{\mathrm{O}^+_{2n}} \simeq \mathcal{P}_{\mathrm{O}^+_{2n}} = \mathcal{P}_2(n) \bigsqcup_{2\leq t,\ t \text{ even},\ t^2\leq n} (\mathcal{P}_2(n-t^2)^{\mathrm{I}} \sqcup \mathcal{P}_2(n-t^2)^{\mathrm{II}})$$

$$\mathcal{E}(\mathrm{O}^-_{2n})_1 \simeq \mathcal{S}_{\mathrm{O}^-_{2n}} \simeq \mathcal{P}_{\mathrm{O}^-_{2n}} = \bigsqcup_{1\leq t,\ t \text{ odd},\ t^2\leq n} (\mathcal{P}_2(n-t^2)^{\mathrm{I}} \sqcup \mathcal{P}_2(n-t^2)^{\mathrm{II}})$$

Example: $\mathcal{E}(\mathrm{Sp}_4(q))_1$

$\mathcal{E}(\mathrm{Sp}_4(q))_1$	dimension	$\mathcal{S}_{\mathrm{Sp}_4}$	$\mathcal{P}_{\mathrm{Sp}_4}$	cuspidality
θ_{14}	1	$\binom{2}{-}$	$\begin{bmatrix} 2 \\ - \end{bmatrix}$	
θ_{11}	$\frac{1}{2}q(q^2+1)$	$\binom{2,1}{0}$	$\begin{bmatrix} 1,1 \\ - \end{bmatrix}$	
θ_9	$\frac{1}{2}q(q+1)^2$	$\binom{2,0}{1}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	
θ_{12}	$\frac{1}{2}q(q^2+1)$	$\binom{1,0}{2}$	$\begin{bmatrix} - \\ 2 \end{bmatrix}$	
θ_{10}	$\frac{1}{2}q(q-1)^2$	$\binom{-}{2,1,0}$	$\begin{bmatrix} - \\ - \end{bmatrix}$	Yes
θ_{13}	q^4	$\binom{2,1,0}{2,1}$	$\begin{bmatrix} - \\ 1,1 \end{bmatrix}$	

$$\mathcal{P}_{\mathrm{Sp}_4} = \mathcal{P}_2(2) \sqcup \mathcal{P}_2(0)$$

Example: $\mathcal{E}(\mathrm{O}_4^+(q))_1$

$\mathcal{E}(\mathrm{O}_4^+(q))_1$	dimension	$\mathcal{S}_{\mathrm{O}_4^+}$	$\mathcal{P}_{\mathrm{O}_4^+}$
$\chi_1, \chi_1 \cdot \mathrm{sgn}$	1	$\binom{2}{0}, \binom{0}{2}$	$\begin{bmatrix} 2 \\ - \end{bmatrix}, \begin{bmatrix} - \\ 2 \end{bmatrix}$
$\chi_2, \chi_2 \cdot \mathrm{sgn}$	q^2	$\binom{2,1}{1,0}, \binom{1,0}{2,1}$	$\begin{bmatrix} 1,1 \\ - \end{bmatrix}, \begin{bmatrix} - \\ 1,1 \end{bmatrix}$
χ_3	$2q$	$\binom{1}{1}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathcal{P}_{\mathrm{O}_4^+} = \mathcal{P}_2(2)$

Orthogonal basis for $\mathcal{V}(G)^\sharp_1$

- for an irreducible character χ of W_n, define

$$R^{\mathrm{Sp}_{2n}}_\chi = \frac{1}{|W_n|}\sum_{w\in W_n}\chi(w)R^{\mathrm{Sp}_{2n}}_{\mathbf{T}_w,\mathbf{1}};$$

$$R^{\mathrm{O}^\epsilon_{2n}}_\chi = \frac{1}{|W^\epsilon_n|}\sum_{w\in W^\epsilon_n}\chi(w)R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}_w,\mathbf{1}}$$

  where $W_n = W^+_n \sqcup W^-_n$
- we define $R^{\mathrm{Sp}_{2n}}_\Sigma$ for $\Sigma \in \mathcal{S}_{n,1} \simeq \mathcal{P}_2(n) \simeq \mathcal{E}(W_n)$
- we define $R^{\mathrm{O}^\epsilon_{2n}}_\Sigma$ for $\Sigma \in \mathcal{S}_{n,0} \simeq \mathcal{P}_2(n) \simeq \mathcal{E}(W_n)$
- for $\Sigma \in \mathcal{S}_{n,0}$, we have $R^{\mathrm{O}^\epsilon_{2n}}_{\Sigma^{\mathrm{t}}} = \epsilon R^{\mathrm{O}^\epsilon_{2n}}_\Sigma$
- $\{\, R^{\mathrm{Sp}_{2n}}_\Sigma \mid \Sigma \in \mathcal{S}_{n,1}\,\}$ an orthonormal basis for $\mathcal{V}(\mathrm{Sp}_{2n})^\sharp_1$;
- $\{\, R^{\mathrm{O}^\epsilon_{2n}}_\Sigma \mid \Sigma \in \mathcal{S}_{n,0}/\{\Sigma,\Sigma^{\mathrm{t}}\}\,\}$ an orthogonal basis for $\mathcal{V}(\mathrm{O}^\epsilon_{2n})^\sharp_1$ such that

$$\langle R^{\mathrm{O}^\epsilon_{2n}}_\Sigma, R^{\mathrm{O}^\epsilon_{2n}}_\Sigma\rangle = \begin{cases} 1, & \text{if } \Sigma = \Sigma^{\mathrm{t}}; \\ 2, & \text{otherwise.}\end{cases}$$

Decomposition of $\omega^\sharp_{\mathbf{G},\mathbf{G}',1}$

$$\omega^\sharp_{\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'},1} = \frac{1}{2}\sum_{k=0}^{\min(n,n')}\sum_{\chi\in\mathcal{E}(W_k)} R^{\mathrm{Sp}_{2n}}_{I_{n,k}(\chi)} \otimes R^{\mathrm{O}^\epsilon_{2n'}}_{I_{n',k}(\varepsilon_0\chi)}$$

- $I_{n,k}(\chi) = \mathrm{Ind}^{W_n}_{W_k\times W_{n-k}}(\chi\otimes\mathbf{1})$,
- $I_{n',k}(\varepsilon_0\chi) = \mathrm{Ind}^{W_{n'}}_{W_k\times W_{n'-k}}(\varepsilon_0\chi\otimes\mathbf{1})$,
- $\varepsilon_0\colon \mathcal{E}(W_k)\to\{\pm1\}$ with kernel W_k^+

$\mathcal{D}_{\mathbf{G},\mathbf{G}'}$

$$\omega^{\sharp}_{\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'},1} = \frac{1}{2}\sum_{(\Sigma,\Sigma')\in\mathcal{D}_{\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'}}} R^{\mathrm{Sp}_{2n}}_{\Sigma}\otimes R^{\mathrm{O}^{\epsilon}_{2n'}}_{\Sigma'}.$$

- $\Upsilon(\Sigma)=\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ and $\Upsilon(\Sigma')=\begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}$
- $\mathcal{D}_{\mathrm{Sp}_{2n},\mathrm{O}^{+}_{2n'}}=\{\,(\Sigma,\Sigma')\in\mathcal{S}_{n,1}\times\mathcal{S}_{n',0}\mid \mu\preccurlyeq\lambda',\mu'\preccurlyeq\lambda\,\}$
- $\mathcal{D}_{\mathrm{Sp}_{2n},\mathrm{O}^{-}_{2n'}}=\{\,(\Sigma,\Sigma')\in\mathcal{S}_{n,1}\times\mathcal{S}_{n',0}\mid \lambda'\preccurlyeq\mu,\lambda\preccurlyeq\mu'\,\}$
- $\lambda\preccurlyeq\mu$ means $\mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots$ where $\lambda=[\lambda_1,\lambda_2,\ldots]$ and $\mu=[\mu_1,\mu_2,\ldots]$

$\mathcal{B}_{\mathbf{G},\mathbf{G}'}$

let $\mathcal{B}_{\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'}}$ be the subset of $\mathcal{S}_{\mathrm{Sp}_{2n}} \times \mathcal{S}_{\mathrm{O}^{\epsilon}_{2n'}}$ of elements (Λ, Λ') such that

$$\begin{cases} \mu' \preccurlyeq \lambda,\ \mu \preccurlyeq \lambda',\ \mathrm{def}(\Lambda') = -\mathrm{def}(\Lambda) + 1, & \text{if } \epsilon = +; \\ \lambda \preccurlyeq \mu',\ \lambda' \preccurlyeq \mu,\ \mathrm{def}(\Lambda') = -\mathrm{def}(\Lambda) - 1, & \text{if } \epsilon = - \end{cases}$$

where $\Upsilon(\Lambda) = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$ and $\Upsilon(\Lambda') = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix}$

Theorem (P.)

Let $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}^{\epsilon}_{2n'})$. *Then*

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}} \rho^{\sharp}_{\Lambda} \otimes \rho^{\sharp}_{\Lambda'} = \frac{1}{2} \sum_{(\Sigma,\Sigma')\in\mathcal{D}_{\mathbf{G},\mathbf{G}'}} R^{\mathbf{G}}_{\Sigma} \otimes R^{\mathbf{G}'}_{\Sigma'} = \omega^{\sharp}_{\mathbf{G},\mathbf{G}',1}.$$

Example

$\mathcal{B}_{\mathrm{Sp}_4,\mathrm{O}_4^+}$

$\mathcal{S}_{\mathrm{O}_4^+}$	$\mathcal{S}_{\mathrm{Sp}_4}$	$\binom{2}{-}$ $\begin{bmatrix}2\\-\end{bmatrix}$	$\binom{2,1,0}{2,1}$ $\begin{bmatrix}-\\1,1\end{bmatrix}$	$\binom{2,1}{0}$ $\begin{bmatrix}1,1\\-\end{bmatrix}$	$\binom{2,0}{1}$ $\begin{bmatrix}1\\1\end{bmatrix}$	$\binom{1,0}{2}$ $\begin{bmatrix}-\\2\end{bmatrix}$	$\binom{-}{2,1,0}$ $\begin{bmatrix}-\\-\end{bmatrix}$
$\binom{2}{0}$	$\begin{bmatrix}2\\-\end{bmatrix}$	✓			✓	✓	
$\binom{0}{2}$	$\begin{bmatrix}-\\2\end{bmatrix}$	✓					
$\binom{1}{1}$	$\begin{bmatrix}1\\1\end{bmatrix}$	✓		✓	✓		
$\binom{2,1}{1,0}$	$\begin{bmatrix}1,1\\-\end{bmatrix}$		✓		✓		
$\binom{1,0}{2,1}$	$\begin{bmatrix}-\\1,1\end{bmatrix}$			✓			

Strategy I

- a symbol $Z=\binom{z_1,z_3,\ldots}{z_2,z_4,\ldots}$ of defect $1,0$ is *special* if $z_1\geq z_2\geq z_3\geq z_4\geq\cdots$.
- $\mathcal{S}_Z$: the set of symbols in $\mathcal{S}_{\mathbf{G}}$ with the same entries of a special symbol Z
- for example, if $Z=\binom{2,0}{1}\in\mathcal{S}_{\mathrm{Sp}_4}$, then $\mathcal{S}_Z=\{\binom{2,1}{0},\binom{2,0}{1},\binom{1,0}{2},\binom{-}{2,1,0}\}\subset\mathcal{S}_{\mathrm{Sp}_4}$, then

$$\mathcal{V}(\mathrm{Sp}_4(q))_Z=\mathrm{span}\{\rho_{\binom{2,1}{0}},\rho_{\binom{2,0}{1}},\rho_{\binom{1,0}{2}},\rho_{\binom{-}{2,1,0}}\}\subset\mathcal{V}(\mathrm{Sp}_4(q))_1$$

$$\mathcal{V}(\mathrm{Sp}_4(q))^\sharp_Z=\mathrm{span}\{R^{\mathrm{Sp}_4}_{\binom{2,1}{0}},R^{\mathrm{Sp}_4}_{\binom{2,0}{1}},R^{\mathrm{Sp}_4}_{\binom{1,0}{2}}\}\subset\mathcal{V}(\mathrm{Sp}_4(q))^\sharp_1$$

- define $\mathcal{D}_{Z,Z'}=\mathcal{D}_{\mathbf{G},\mathbf{G}'}\cap(\mathcal{S}_Z\times\mathcal{S}_{Z'})$
- define $\mathcal{B}^\epsilon_{Z,Z'}=\mathcal{B}_{\mathbf{G},\mathbf{G}'}\cap(\mathcal{S}_Z\times\mathcal{S}_{Z'})$
- then

$$\omega^\sharp_{\mathbf{G},\mathbf{G}',1}=\sum_{Z,Z'}\frac{1}{2}\sum_{(\Sigma,\Sigma')\in\mathcal{D}_{Z,Z'}}R^{\mathbf{G}}_\Sigma\otimes R^{\mathbf{G}'}_{\Sigma'}$$

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'}=\sum_{Z,Z'}\sum_{(\Lambda,\Lambda')\in\mathcal{B}^\epsilon_{Z,Z'}}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'}$$

Strategy II

Need to prove

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}^\epsilon_{Z,Z'}} \rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'} = \frac{1}{2}\sum_{(\Sigma,\Sigma')\in\mathcal{D}_{Z,Z'}} R^{\mathbf{G}}_\Sigma\otimes R^{\mathbf{G}'}_{\Sigma'}$$

- $\mathcal{B}^\epsilon_{Z,Z'}=\emptyset$ if and only if $\mathcal{D}_{Z,Z'}=\emptyset$
- for $Z=\binom{2t,2t-2,\ldots,0}{2t-1,2t-3,\ldots,1}\in\mathcal{S}_{\mathrm{Sp}_{2t(t+1)}}$, $Z'=\binom{2t+1,2t-1,\ldots,1}{2t,2t-2,\ldots,0}\in\mathcal{S}_{\mathrm{O}^+_{2(t+1)^2}}$, then

$$\mathcal{D}_{Z,Z'}=\{\,(\Sigma,\theta^\epsilon(\Sigma))\mid\Sigma\in\mathcal{S}_Z,\ \mathrm{def}(\Sigma)=1\,\}$$
$$\mathcal{B}^\epsilon_{Z,Z'}=\{\,(\Lambda,\theta^\epsilon(\Lambda))\mid\Lambda\in\mathcal{S}_Z\,\}$$

  where $\theta^\epsilon(\Lambda)=\Lambda^{\mathrm{t}}\cup\binom{-}{2t+1}$
- by Lusztig's result, $\langle\rho_\Lambda,R^{\mathbf{G}}_\Sigma\rangle$ is known, so $\rho^\sharp_\Lambda$ can be written as a linear combination of $R^{\mathbf{G}}_\Sigma$
- for $Z=\binom{2t,2t-2,\ldots,0}{2t-1,2t-3,\ldots,1}\in\mathcal{S}_{\mathrm{Sp}_{2t(t+1)}}$ and $Z'=\binom{2t-1,2t-3,\ldots,1}{2t-2,2t-4,\ldots,0}\in\mathcal{S}_{\mathrm{O}^+_{2t^2}}$, similar
- general cases can be reduced to the two basic cases via isometries of inner product spaces

Strategy III

- need to recover

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}^\epsilon_{Z,Z'}} \rho_\Lambda\otimes\rho_{\Lambda'}=\omega_{Z,Z'}$$

from

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}^\epsilon_{Z,Z'}} \rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'}=\frac{1}{2}\sum_{(\Sigma,\Sigma')\in\mathcal{D}_{Z,Z'}} R^{\mathbf{G}}_\Sigma\otimes R^{\mathbf{G}'}_{\Sigma'}=\omega^\sharp_{Z,Z'}$$

where $\omega_{Z,Z'}$ is the orthogonal projection of $\omega_{\mathbf{G},\mathbf{G}',1}$ onto $\mathcal{V}(G)_Z\otimes\mathcal{V}(G')_{Z'}$

- write

$$\omega_{Z,Z'}=\sum_{\Lambda\in\mathcal{S}_Z,\ \Lambda'\in\mathcal{S}_{Z'}} m_{\Lambda,\Lambda'}\rho_\Lambda\otimes\rho_{\Lambda'}$$

$$\overline{\omega}_{Z,Z'}=\sum_{(\Lambda,\Lambda')\in\mathcal{B}^\epsilon_{Z,Z'}} \rho_\Lambda\otimes\rho_{\Lambda'}$$

where $m_{\Lambda,\Lambda'}\in\mathbb{N}\cup\{0\}$

- for $f\in\mathcal{V}(G\times G')^\sharp_1$, we have

$$\langle f,\overline{\omega}_{Z,Z'}\rangle=\langle f,\overline{\omega}^\sharp_{Z,Z'}\rangle=\langle f,\omega^\sharp_{Z,Z'}\rangle=\langle f,\omega_{Z,Z'}\rangle$$

Lusztig's theory of "*cells*" provides many uniform f

Lusztig's "cell"

- for a special symbol $Z = \binom{S}{T}$, let $Z_{\rm I} = Z \smallsetminus \binom{S\cap T}{S\cap T}$ denote the subsymbol of "singles"
- for $M \subset Z_{\rm I}$, define $\Lambda_M = (Z \smallsetminus M) \cup M^{\rm t}$, e.g., $\Lambda_\emptyset = Z$, $\Lambda_{Z_{\rm I}} = Z^{\rm t}$
- an *arrangement* of $Z_{\rm I}$ is a partition Φ of $Z_{\rm I}$ into

$$\begin{cases} d \text{ pairs } \binom{s_i}{t_j} \text{ and a single } \{s\}, & \text{if } |Z_{\rm I}| = 2d+1; \\ d \text{ pairs } \binom{s_i}{t_j}, & \text{if } |Z_{\rm I}| = 2d. \end{cases}$$

- we say $\Psi \leq \Phi$ if Ψ is the union of some pairs in Φ.
- for $\Psi \leq \Phi$, we define $C_{\Phi,\Psi} \subset \mathcal{S}_Z$ consisting of Λ_M such that
  - M contains either none or two entries of each pair in Ψ; and
  - M contains exactly one entry of each pair in $\Phi \smallsetminus \Psi$; and
  - $|M|$ even.
- Lusztig shows that $\sum_{\Lambda \in C_{\Phi,\Psi}} \rho_\Lambda \in \mathcal{V}(G)$ is uniform.

An Example

- $Z = Z_{\mathrm{I}} = \binom{2,0}{1} \in \mathcal{S}_{\mathrm{Sp}_4}$,
- $\mathcal{S}_Z = \{\binom{2,1}{0}, \binom{2,0}{1}, \binom{1,0}{2}, \binom{-}{2,1,0}\}$, $\{\, \rho_\Lambda \mid \Lambda \in \mathcal{S}_Z \,\} = \{\theta_{11}, \theta_9, \theta_{12}, \theta_{10}\}$
- $\dim \mathcal{V}(\mathrm{Sp}_4(q))_Z = 4$, $\dim \mathcal{V}(\mathrm{Sp}_4(q))_Z^\sharp = 3$
-

Φ	Ψ	$C_{\Phi,\Psi}$	$\sum_{\Lambda\in C_{\Phi,\Psi}} \rho_\Lambda$
$\binom{2}{1} \cup \{0\}$	$\emptyset$	$\binom{-}{2,1,0}, \binom{2,1}{0}$	$\theta_{10} + \theta_{11}$
	$\binom{2}{1}$	$\binom{2,0}{1}, \binom{1,0}{2}$	$\theta_9 + \theta_{12}$
$\binom{0}{1} \cup \{2\}$	$\emptyset$	$\binom{-}{2,1,0}, \binom{1,0}{2}$	$\theta_{10} + \theta_{12}$
	$\binom{0}{1}$	$\binom{2,0}{1}, \binom{2,1}{0}$	$\theta_9 + \theta_{11}$

A System of Linear Equations

- suppose $Z=\binom{2t,2t-2,\ldots,0}{2t-1,2t-3,\ldots,1}\in\mathcal{S}_{\mathrm{Sp}_{2t(t+1)}}$ and $Z'=\binom{2t+1,2t-1,\ldots,1}{2t,2t-2,\ldots,0}\in\mathcal{S}_{\mathrm{O}^\epsilon_{2(t+1)^2}}$
- for $\Psi\leq\Phi$ and $\Psi'\leq\Phi'$ of Z_{I}, define

$$f=\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\theta(\Phi'),\theta(\Psi')}}\rho_\Lambda\otimes\rho_{\Lambda'}\in\mathcal{V}(G\times G')^\sharp_1$$

$$x_{\Lambda,\Lambda_1}=\langle\rho_\Lambda\otimes(\rho_{\theta^\epsilon(\Lambda_1)}+\rho_{\theta^\epsilon(\Lambda_1)^{\mathrm{t}}}),\omega_{Z,Z'}\rangle=m_{\Lambda,\theta^\epsilon(\Lambda_1)}+m_{\Lambda,\theta^\epsilon(\Lambda_1)^{\mathrm{t}}}$$

- $\langle f,\omega_{Z,Z'}\rangle=\langle f,\overline{\omega}_{Z,Z'}\rangle$ is reduced to

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda_1\in C_{\Phi',\Psi'}}x_{\Lambda,\Lambda_1}=|C_{\Phi,\Psi}\cap C_{\Phi',\Psi'}|$$

- for $\Lambda\neq\Lambda_1$, there are Φ,Ψ,Φ',Ψ' such that $\Lambda\in C_{\Phi,\Psi}$, $\Lambda_1\in C_{\Phi',\Psi'}$ and $C_{\Phi,\Psi}\cap C_{\Phi',\Psi'}=\emptyset$, this concludes that $x_{\Lambda,\Lambda_1}=0$
- for Λ, there are Φ,Ψ,Φ',Ψ' such that $C_{\Phi,\Psi}\cap C_{\Phi',\Psi'}=\{\Lambda\}$, this concludes that $x_{\Lambda,\Lambda}=1$

The Main Result

- then we conclude that the only non-negative integral solution of the system of linear equations is

$$m_{\Lambda,\Lambda'}+m_{\Lambda,\Lambda'^{\mathrm{t}}}=\begin{cases}1, & \text{if } (\Lambda,\Lambda') \text{ or } (\Lambda,\Lambda'^{\mathrm{t}})\in\mathcal{B}^{\epsilon}_{Z,Z'};\\ 0, & \text{otherwise}\end{cases}$$

- the Howe correspondence is compatible with the parabolic induction, we finally conclude that $m_{\Lambda,\Lambda'}=1$ if and only if $(\Lambda,\Lambda')\in\mathcal{B}^{\epsilon}_{Z,Z'}$, i.e, $\omega_{Z,Z'}=\overline{\omega}_{Z,Z'}$
- general cases can be reduced to two basic cases

Theorem (P.)

Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'})$. *Then*

$$\omega_{\mathbf{G},\mathbf{G}',1}=\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}}\rho_\Lambda\otimes\rho_{\Lambda'}.$$

Application I: extension of η-correspondence

Theorem (Gurevich-Howe 2017)

Let $(\mathbf{G},\mathbf{G}') = (\mathrm{O}_k^\epsilon, \mathrm{Sp}_n)$. *Suppose* $k \leq \frac{n}{2}$ *(i.e., in stable range). Then for each* $\rho \in \mathcal{E}(G)$, *there is a unique* $\eta(\rho) \in \Theta_{\mathbf{G}'}(\rho)$ *of* Θ*-rank* k *and all other elements in* $\Theta_{\mathbf{G}'}(\rho)$ *are of* Θ*-rank less than* k. *The mapping* $\eta\colon \mathcal{E}(\mathrm{O}_k^\epsilon) \to \mathcal{E}(\mathrm{Sp}_n)$ *is injective.*

- for $\rho' \in \mathcal{E}(G'_n)$, let $\Theta\text{-rk}(\rho')$ denote the minimum of k such that $\rho'\chi$ occurs in $\Theta_{\mathbf{G}_k,\mathbf{G}'_n}$ for some linear character $\chi \in \mathcal{E}(G'_n)$ and some $\mathbf{G}_k$

Theorem (P.)

Let $(\mathbf{G},\mathbf{G}') = (\mathrm{O}_k^\epsilon, \mathrm{Sp}_n)$. *There exists a maximal one-to-one sub-correspondence* $\underline{\theta}$ *of* $\Theta_{\mathbf{G},\mathbf{G}'}$ *which is symmetric, semi-persistent, and compatible with the Lusztig correspondence. Moreover,* $\underline{\theta} = \eta$ *if* $(\mathbf{G},\mathbf{G}')$ *is in stable range.*

Application II: Degree difference in the η-correspondence

- let $\mathbf{G}_n$ be $\mathrm{Sp}_n, \mathrm{O}_n^\epsilon, \mathrm{U}_n$
- the dimension $\rho(1)$ is a polynomial in q, let $\deg_q(\rho)$ denote the degree of this polynomial

Theorem (P.)

Let $\rho' \in \mathcal{E}(G'_n)$. Suppose that $\Theta\text{-}\mathrm{rk}(\rho') = k$. Then there exist a linear character $\chi \in \mathcal{E}(G'_n)$, $\mathbf{G}_k$ and $\rho \in \mathcal{E}(G_k)$ such that $\underline{\theta}(\rho) = \rho'\chi$ for $(\mathbf{G}_k, \mathbf{G}'_n)$. Moreover,

$$\deg_q(\rho') = \deg_q(\rho) + \begin{cases} \frac{1}{2}k(n-k+1), & \text{if } \mathbf{G}'_n = \mathrm{Sp}_n; \\ \frac{1}{2}k(n-k-1), & \text{if } \mathbf{G}'_n = \mathrm{O}_n^\epsilon; \\ k(n-k), & \text{if } \mathbf{G}'_n = \mathrm{U}_n. \end{cases}$$

- The theorem is known by Gurevich-Howe for $(\mathrm{O}_k^\epsilon, \mathrm{Sp}_n)$ in stable range.

The End. Thank You.