Kac-Moody superalgebras and Duflo-Serganova functors

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Lie superalgebras: definition and first examples

The base field is $\mathbb{C}$.
$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$; parity: $p(x) = \bar{i}$ for $x \in \mathfrak{g}_{\bar{i}}$.
Axioms: anticommutativity and Jacobi identity:
$[a, b] + (-1)^{p(a)p(b)}[b, a] = 0$;
$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]$.
$\mathfrak{g}_{\bar{0}}$ is a Lie algebra; $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module.

$\mathfrak{gl}(m|n)$: $(m+n) \times (m+n)$ block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad p(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}) = \bar{0}, \quad p(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}) = \bar{1}$$

$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) | \ Tr\, A = Tr\, D\}$, $\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/\mathbb{C}Id$.

“Nice superalgebras” vs. semisimple Lie algebras

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For (FIN) $\mathfrak{g}_0$ is reductive and

$$\mathfrak{gl}(m|0) = \mathfrak{gl}_m, \ \ \mathfrak{osp}(M|0) = \mathfrak{o}_M, \ \ \mathfrak{osp}(0|N) = \mathfrak{sp}_N.$$

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By contrast with semisimple Lie algebras, fin.-dim. modules are not completely reducible and the characters are not given by Weyl character formula. These works only for so-called typical modules.

Iso-sets

Let $\mathfrak{g}$ be any superalgebra which contains a max. fin.-dim. commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ which acts diagonally in the adjoint representation of $\mathfrak{g}$.
The multisets of even and odd roots: $\Delta_0, \Delta_1 \subset \mathfrak{h}^* \setminus \{0\}$.
We write each $a \in \mathfrak{g}_i$ (for $i = 0, 1$) in the form

$$a = \sum_{\alpha \in \text{supp}(a)} a_\alpha, \quad \text{where } a_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}, \ \text{supp}(a) \subset \Delta_i \cup \{0\}.$$

Definition We say that $S \subset \Delta_1$ is an iso-set if the elements of S are linearly independent and for each $\alpha, \beta \in \Delta_1 \cap (S \cup (-S))$ one has $\alpha + \beta \notin \Delta_0$.
For (FIN): $S \subset \Delta_1$ is an iso-set iff $(S|S) = 0$ and S is linearly independent;
For (AFF): $S \subset \Delta_1$ is an iso-set iff $(S|S) = 0$ and S is linearly independent modulo $\mathbb{C}\delta$, where δ is the minimal imaginary root.

Iso-sets and defect

Example: $\mathfrak{g} := \mathfrak{gl}(m|n)$, $\mathfrak{g}_{\bar{1}} = \mathfrak{gl}_m \times \mathfrak{gl}_n$ with

$$\Delta(\mathfrak{gl}_m) = \{\varepsilon_i - \varepsilon_j\},\ \ \Delta(\mathfrak{gl}_n) = \{\delta_i - \delta_j\},\ \ \Delta(\mathfrak{g}_{\bar{1}} = \{\pm(\varepsilon_i - \delta_j)\}.$$

The form: $(\varepsilon_i|\varepsilon_j) = -(\delta_i|\delta_j) = \delta_{ij}$.
$S_s := \{\varepsilon_i - \delta_i\}_{i=1}^{s}$; $S_{\min(m,n)}$ is maximal.

For (FIN), (AFF):
The defect of $\mathfrak{g}$ is the cardinality of the maximal iso-set.

Example: defect of $\mathfrak{gl}(m|n)$ is $\min(m, n)$.
Remark: for the “strange” superalgebras $\mathfrak{p}_n, \mathfrak{q}_n$ the above definition give *defect*$(\mathfrak{p}_n) = n$ and *defect*$(\mathfrak{q}_n) = [\frac{n}{2}]$.

Iso-sets and blocks in $\mathcal{O}$

Theorem. For (FIN) and $\mathfrak{q}_n$ the modules $L(\lambda-\rho)$, $L(\nu-\rho)$ lie in the same block in $\mathcal{O}(\mathfrak{g})$ if and only if $\nu \in W_\lambda(\lambda+\mathbb{Z}S_\lambda)$, where S_λ is a maximal iso-set S_λ orthogonal to λ.
For (AFF) the same holds if $L(\lambda-\rho)$ is non-critical ($(\lambda|\delta)\neq 0$).

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(FIN)+$\mathfrak{q}_n$: The Dulfo-Musson Theorem allows to extend the notion of atypicality to central characters and thus to all simple $\mathfrak{g}$-modules.

Cores

Notation: $\lambda \sim \nu$ if $L(\lambda-\rho), L(\nu-\rho)$ have the same central character.

For s/s Lie algebras $HC: Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ gives $\lambda' \sim \lambda$ iff $\lambda' \in W\lambda$. Writing $\lambda = \sum_{i=1}^m a_i\varepsilon_i$ we have $\lambda \sim \lambda'$ iff $\mathfrak{gl}_m$: $\{a_i\}_{i=1}^m = \{a'_i\}_{i=1}^m$, $\mathfrak{o}_{2m+1}, \mathfrak{sp}_{2m}$: $\{|a_i|\}_{i=1}^m = \{|a'_i|\}_{i=1}^m$.

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$\mathfrak{gl}(m|n)$: $\lambda = \sum_{i=1}^m a_i\varepsilon_i - \sum_{i=1}^n \delta_i$
Let $Core(\lambda)$ be the multiset obtained from $\{a_i\}_{i=1}^m \coprod \{b_j\}_{j=1}^n$ by deleting the maximal number of pairs satisfying $a_i = b_j$.
Example: $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1 - 2\delta_2$ $Core(\lambda) = \{1, 1\} \coprod \{2\}$.
Then $\lambda \sim \lambda'$ iff $Core(\lambda) = Core(\lambda')$.

Cores for (AFF)

$\mathfrak{gl}(m|n)^{(1)}$: Set $k := (\lambda|\delta)$. Let *Core*(λ) be the multiset obtained from $\{a_i\}_{i=1}^m \coprod \{b_j\}_{j=1}^n$ by deleting the maximal number of pairs satisfying $a_i - b_j \in \mathbb{Z}k$. We view the elements of the multiset *Core*(λ) as elements in $\mathbb{C}/\mathbb{Z}k$.

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Slightly more complicated formulae for $\mathfrak{osp}(M|N)^{(1)}$ and the twisted cases.

Theorem. (G., arXiv: 2010.05721) $L(\nu-\rho), L(\lambda-\rho)$ are in the same (non-critical) block in $\mathcal{O}$, then *Core*$(\lambda) =$ *Core*(ν).

Duflo-Serganova functors

The Dulfo-Serganova functors are tensor functors relating representations of different Lie superalgebras.
These functor were introduced by Duflo and Serganova in "On associated variety for Lie superalgebras", arXiv:0507198; they studied these functors for (FIN). DS_x

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PRESERVES

- the dual Coxeter number and the type of the algebra: $DS_x(\mathfrak{gl}(m|n)) = \mathfrak{gl}(m - rank\ x|n - rank\ x)$, etc.
- the core of a highest weight module and of the central character (for the non-exceptional algebras).

Construction and first properties

Set $X(\mathfrak{g}) := \{x \in \mathfrak{g}_1 | \, [x, x] = 0\}$.
Definition: For $x \in X(\mathfrak{g})$ we set $\mathrm{DS}_x(M) := M^x/xM$. Then $\mathfrak{g}_x := \mathrm{DS}_x(\mathfrak{g}) = \mathfrak{g}^x/[x, \mathfrak{g}]$ is a Lie superalgebra and

$$\mathrm{DS}_x : M \mapsto \mathrm{DS}_x(M)$$

is a functor from the category of $\mathfrak{g}$-modules to the category of $\mathrm{DS}_x(\mathfrak{g})$-modules.

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$\mathrm{DS}_x(\mathfrak{gl}(m|n)) \cong \mathfrak{gl}(m-r|n-r), \; \mathrm{DS}_x(\mathfrak{q}_n) \cong \mathfrak{q}_{n-2r};$
$\mathrm{DS}_x(\mathfrak{osp}(M|N)) \cong \mathfrak{osp}(M-2r|N-2r),$
$\mathrm{DS}_x(D(2|1, a) = \mathbb{C}, \; \mathrm{DS}_x(G(3)) = \mathfrak{sl}_2, \; \mathrm{DS}_x(F(4)) = \mathfrak{sl}_3$ for $x \neq 0$.

$X_{iso}(\mathfrak{g})$

The situation is more complicated for affine case.
Example: $\mathfrak{g} = \mathfrak{sl}(2|1)^{(1)}$

$$x \in \mathfrak{gl}_\alpha \text{ for odd } \alpha \quad \mathrm{DS}_x(\mathfrak{g}) = \mathbb{C}K \times \mathbb{C}d,$$

If $y := x + xt$, then y is odd, $y^2 = 0$ and
$\mathrm{DS}_y(\mathfrak{g}) = \mathbb{C}K \times \mathfrak{t}, \;\; \mathfrak{t}_0 = span(h, e), \;\; \mathfrak{t}_1 = spanF$

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We set $X_{iso}(\mathfrak{g}) := \{x \in \mathfrak{g} | \text{ supp}(x) \text{ is an iso-set}\}$.
Facts. $X_{iso}(\mathfrak{g}) \subset X(\mathfrak{g})$;
$X(\mathfrak{g}) = X_{iso}(\mathfrak{g})$ if $\mathfrak{g}$ is a fin.-dim. KM or $\mathfrak{p}_n, \mathfrak{q}_n, \mathfrak{sl}(n|n)$.
$X(\mathfrak{g}) \neq X_{iso}(\mathfrak{g})$ if $\mathfrak{g}$ is affine or $\mathfrak{g} = \mathfrak{pgl}(n|n), \mathfrak{psl}(n|n), \mathfrak{pq}_n$ etc.

Depth

We introduce $\mathrm{depth}(\mathfrak{g}) \in \mathbb{N} \cup \{\infty\}$ by the formula

$$\mathrm{depth}(\mathfrak{g}) = \begin{cases} 0 & \text{if } X_{iso}(\mathfrak{g}) = 0 \\ 1 + \max\limits_{x \in X_{iso} \setminus \{0\}} \mathrm{depth}(\mathfrak{g}_x) & \text{if } X_{iso}(\mathfrak{g}) \neq \emptyset. \end{cases}$$

For $x \in X$ we define $\mathrm{rank}\ x := \mathrm{depth}(\mathfrak{g}) - \mathrm{depth}(\mathfrak{g}_x)$ and then introduce $\mathrm{depth}(N)$ in a similar fashion; for a full subcategory of $\mathfrak{g}$-modules $\mathcal{C}$ we define $\mathrm{depth}(\mathcal{C})$ as the maximum of $\mathrm{depth}(N)$ for $N \in \mathcal{C}$.

One has $\mathrm{depth}\,\mathfrak{g} \geq \mathrm{defect}\,\mathfrak{g}$ where defect is the maximal cardinality of an iso-set.

$$\mathrm{depth}(N' \oplus N'') = \max(\mathrm{depth}(N')), \mathrm{depth}(N'')),$$
$$\mathrm{depth}(N' \otimes N'') = \min(\mathrm{depth}(N')), \mathrm{depth}(N'')).$$

Propeties of depth

If supp(x) is an iso-set of cardinality r, then rank $x \geq r$;
for (FIN), (AFF) and $\mathfrak{q}_n$: rank $x = r$ and depth $\mathfrak{g} =$ *defect* $\mathfrak{g}$.

Example. For (FIN)+$\mathfrak{q}_n$, (AFF) or $\mathfrak{q}_n$: depth of a block in $\mathcal{O}(\mathfrak{g})$ is equal to the atypicality. This allows to define atypicality for other modules in (AFF).

Theorem (Serganova, 2011) if $\mathfrak{g}$ is (FIN) and L is a fin.-dim. simple module, then depth(L) $=$ *atyp*(L).

This does not hold for $\mathfrak{q}_n$, but the depth of a block in *Fin*($\mathfrak{q}_n$) is equal to the atypicality.

DS_x and the central characters

Let $\mathfrak{g}$ be any fin.-dim. superalgebra and $\mathfrak{g}_x := \mathrm{DS}_x(\mathfrak{g})$. The map

$$U(\mathfrak{g})^{\mathrm{ad}\,x} \to U(\mathfrak{g})^{\mathrm{ad}\,x}/[x, U(\mathfrak{g})] = \mathrm{DS}_x(U(\mathfrak{g})) = U(\mathfrak{g}_x)$$

induces an algebra homomorphism

$$\theta_x : Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \to U(\mathfrak{g}_x)^{\mathrm{ad}\,\mathfrak{g}_x} = Z(\mathfrak{g}_x).$$

The equality of the dual Coxeter numbers follows from $\theta_x(Cas(\mathfrak{g})) = Cas(\mathfrak{g}_x)$.
If N is a $\mathfrak{g}$-module with the central character χ, then $\mathrm{DS}_x(N)$ is a $\mathfrak{g}_x$-module with the central characters in $(\theta_x^*)^{-1}(\chi)$.

Corollary.

View N as a $\mathfrak{g}^x$-module (or as a $\mathfrak{g}_x$-module if $\mathfrak{g}_x$ "can be embedded to $\mathfrak{g}''$). If L' is a simple $\mathfrak{g}^x$ (resp., $\mathfrak{g}_x$)-module with the central character not in $(\theta_x^*)^{-1}(\chi)$, then

$$[N : L'] = [N : \Pi(L')].$$

Proof. We have $\mathfrak{g}^x$-isomorphisms $N^x/xN \cong \mathrm{DS}_x(N)$ and $N/N^x \cong \Pi(xN)$ (given by the action of x). Hence in the Grothedieck group of $\mathfrak{g}^x$-modules

$$[N] = [N^x] + [\Pi(xN)] = [\mathrm{DS}_x(N)] + +[xN] + [\Pi(xN)]$$

which gives

$$[N : L'] - [N : \Pi(L')] = [\mathrm{DS}_x(N) : L'] - [\mathrm{DS}_x(N) : \Pi(L')].$$

DS_x and cores for (AFF)

Take $x \in X(\mathfrak{g})$ such that $\mathrm{supp}(x) \subset (-\Sigma \cup \Sigma)$ (then $x \in X_{iso}$). Then $\mathfrak{g}_x$ is "of the same type" as $\mathfrak{g}$ with $h^\vee(\mathfrak{g}) = h^\vee(\mathfrak{g}_x)$.

Theorem (G., arXiv: 2010.05721)

Assume that $L(\lambda - \rho)$ is "non-critical" (i.e., $(\lambda|\delta) \neq 0$) and $[\mathrm{DS}_x(L(\lambda - \rho)) : L_{\mathfrak{g}_x}(\nu - \rho)] \neq 0$.

- "DS_x reduces the atypicality by rank x": for λ, ν as above, $atyp\ \nu = atyp\ \lambda - r$;
- For the non-exceptional cases $Core(\lambda) = Core(\nu)$.

DS_x and the central characters for (FIN)

Duflo-Serganova results for $\theta_x : Z(\mathfrak{g}) \to Z(\mathfrak{g}_x)$ (for (FIN)+$\mathfrak{q}_n$):

- θ_x is surjective for $\mathfrak{g} \neq \mathfrak{osp}(2m|2n), D(2|1,a), F(4)$; *Im* $\theta_x = Z(\mathfrak{g}_x)^\sigma$, for an outer involution σ of $\mathfrak{g}_x$ in the remaining cases.

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- The map θ_x^* increases atypicality by rank x, so DS_x reduces the atypicality by rank x;
- the image of θ_x^* consists of the central charatcers of atypicality $\geq$ rank x;

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- the image of θ_x^* consists of the central charatcers of atypicality $\geq$ rank x;
- the fibers of θ_x^* coincide are the σ-orbits in $SpecZ(\mathfrak{g}_x)$;
- the map θ_x^* preserves the cores of central characters.

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- For non-exceptional cases these multiplicities are given in terms of so-called “arc diagrams”.

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- The multiplicities of irreducibles in $\mathrm{DS}_1(L)$ are at most 2 (at most 1 for type I)
- For non-exceptional cases these multiplicities are given in terms of so-called "arc diagrams".
- For $\mathfrak{g} \neq \mathfrak{p}_n$: if $L' \ncong L''$ are subquotients of $\mathrm{DS}_1(L)$, then $\mathrm{Ext}^1(L', L'') = 0$.
- For $\mathfrak{g} \neq \mathfrak{p}_n, \mathfrak{q}_n$: $\mathrm{DS}_x(L)$ is completely reducible and $\mathrm{DS}_1(\mathrm{DS}_1(\ldots(\mathrm{DS}_1(L\ldots)) \cong \mathrm{DS}_s(L)$.

Similar results for the integrable $\mathfrak{gl}(1|n)^{(1)}$-modules in
M. Gorelik, V. Serganova, Comm. Math. Phys. **364** (2018).

Example

Let N be a $\mathfrak{g}$-module and L' be a simple $\mathfrak{g}_x$-module.
By above, $|[N : L'] - [N : \Pi(L')]| \leq 2$ and $= 0$ if
$atypL' \neq atypN - rankx$ or $core(L') \neq core(N)$.

For instance, for a typical $\mathfrak{gl}(m|n)$-module N
$[N : L'] = [N : \Pi(L')]$ for each $\mathfrak{gl}(m-1|n-1)$-module L' (for a "special" copies of $\mathfrak{gl}(m-1|n-1)$ in $\mathfrak{gl}(m|n)$).

DS for the affine vertex algebras

M. Gorelik, V. Serganova, On DS-functor for affine Lie superalgebras, arXiv:1711.10149.