Quantization and Duality for Spherical Varieties

David Ben-Zvi
University of Texas at Austin

Lie Groups Seminar
MIT
May 2021

Overview

Geometric quantization: representation theory as quantum mechanics of hamiltonian actions.

Joint work with Yiannis Sakellaridis and Akshay Venkatesh: recast relative Langlands program as duality between "arithmetic" and "spectral" forms of higher[1] geometric quantization

[1] in sense of higher-dimensional quantum field theory

Setting: Spherical varieties

Spherical variety: nonabelian version of toric variety

G reductive, split$/k$.
$G \circlearrowright X$ (normal, affine) is a spherical variety if Borel $B \subset G$ has an open orbit

- Tate: Toric varieties
- Hecke: $PGL_2/\mathbb{G}_m$
- Eisenstein: Flag varieties G/P (or G/U as $G \times L$-space)
- Symmetric spaces G/K
- Group: $G = H \times H \circlearrowright X = H$
- Branching, Gan-Gross-Prasad : $GL_{n+1} \times GL_n \circlearrowright GL_{n+1}$, $SO_{2n+1} \times SO_{2n} \circlearrowright SO_{2n+1}, \cdots$

Hyperspherical varieties

Microlocal perspective: focus on $M = T^*X \xrightarrow{\mu} \mathfrak{g}^*$ not X.

affine Hamiltonian variety $G \circlearrowright M \to \mathfrak{g}^*$ is hyperspherical[2] if $\mathcal{O}(M)^G$ is Poisson commutative.

Require $G \circlearrowright M$ *graded* – equipped with commuting $\mathbb{G}_m$ action of weight 2 on $\check{M} \to \mathfrak{g}^*$.

- Cotangents $M = T^*X$ to spherical varieties
- Hypertoric varieties
- Whittaker: $T^*G//_\psi N$
- Θ-correspondence: $SO_{2n} \times Sp_{2n} \circlearrowright Std \otimes Std$.

[2] multiplicity one / coisotropic

Dual data for spherical varieties

X spherical. Assume smooth, affine[3].

Extract increasingly rich algebraic/combinatorial data:[4]

- A_X torus
- W_X little Weyl group
- $\check{G}_X \to \check{G}$ dual group
- $\check{G}_X \times SL_2 \to \check{G}$ Arthur parameter

Strongly Tempered case: $\check{G}_X = \check{G}$.

e.g. Tate, Hecke, Whittaker, Rankin-Selberg, Gan-Gross-Prasad, ...

[3] Also: no roots of type N

[4] Luna, Vust, Brion, Knop, Gaitsgory, Nadler, Sakellaridis, Venkatesh,...

Knop geometry I

F. Knop: the dual data controls Hamiltonian flows on M

• Invariant moment map

$$M \xrightarrow{\mu} \mathfrak{g}^* \longrightarrow \mathfrak{c} = \mathfrak{h}^*/W$$

factors through

$$M//G \xrightarrow{\sim} \mathfrak{c}_X = \mathfrak{a}_X^*/W_X$$

• Harish-Chandra homomorphism $Z(U\mathfrak{g}) \to \mathcal{D}(X)$ lifts to isomorphism

$$Z(U\mathfrak{g}_X) := Sym(\mathfrak{a}_X)^{W_X} \xrightarrow{\sim} \mathcal{D}(X)^G$$

Knop geometry II

Much stronger: invariant hamiltonians integrate to action of

$$J_X \circlearrowright M, \qquad J_X \longrightarrow \mathfrak{c}_X \xleftarrow{\ \mu\ } M$$

Kostant-Toda lattice in $G_X := (\check{G}_X)^\vee$
(group-scheme of regular centralizers)

$\rightsquigarrow$ complete birational[5] description of M

Suggestive: $G \circlearrowright X$ looks like "Langlands lifting" of strongly tempered G_X-variety associated to dual data $\check{G}_X \to \check{G}$..

[5] BZSV: extend off codimension 2. With Gunningham: conjecture quantum version of Knop integration

The critical representation

One more crucial ingredient to describe spherical varieties:

- $\check{G}_X \circlearrowright S_X$ symplectic representation

In strongly tempered case S_X is all the data we have.

- Tate: $S_X = T^*\mathbb{A}^1$
- Hecke: $S_X = T^*Std$
- Whittaker: $S_X = 0$

Weights of S_X (or variant $V_X = S_X \oplus \check{\mathfrak{g}}^e/\check{\mathfrak{g}}_X$) come from Sakellaridis' Plancherel formula;
geometry: see [Sakellaridis-Wang]

Sakellaridis-Venkatesh on Relative Langlands Program

Sakellaridis-Venkatesh: dual data of X controls local harmonic analysis on $X(K_v)$ and global theory of X-periods of automorphic forms.

Useful to organize the questions into automorphic quantization $\Theta_M \in \mathcal{A}_G$ of $G \circlearrowright M$ in a variety of settings:

	local	global
geometric	$\overline{\mathbb{F}}_q((t))$ or $\mathbb{C}((t))$	C curve$/\overline{\mathbb{F}}_q$ or $\mathbb{C}$
arithmetic	$\mathbb{F}_q((t))$	C curve$/\mathbb{F}_q$

Automorphic Quantization: Local

To $G \circlearrowright M$ hamiltonian [e.g. $M = T^*X$ for $G \circlearrowright X$] seek to attach:

- K_v local nonarchimedean: unitary representation $\Theta_M(v)$ of $G(K_v)$ quantizing $M(K_v)$

$$L^2(X(K))$$

- Basic spherical vector $\Phi_M(v) \in \Theta_M(v)$

$$1_{X(O_v)}$$

Automorphic Quantization: Global

- F global field: theta series, a $G(\mathbb{A}_F)$-intertwiner

$$\Theta_M(F) : \bigotimes_v{}' \Theta_M(v) \longrightarrow C^\infty(G(F)\backslash G(\mathbb{A}_F))$$

$$\sum_{\gamma \in X(F)} \Phi(\gamma \cdot g)$$

Function fields: X-theta series at a G-bundle counts sections of associated X-bundle,

i.e. pushforward along

$$Bun_{G,X}(C) = \{G\text{-bundle} + \text{section of } X\text{-bundle}\} \xrightarrow{\quad\pi\quad} Bun_G(C)$$

Automorphic Quantization: Geometric

Extend to geometric setting by function-sheaf dictionary (only unramified today, and assume $M = T^*X$ polarized)

- Local geometric:

$$\underline{Sph} = Shv(G(O)\backslash G(K)/G(O)) \circlearrowright Shv(G(O)\backslash X(K))$$

with basic sheaf $\underline{\Phi}_X = \underline{k}_{G(O)\backslash X(O)}$

- Global geometric:

$$\Theta_X = \pi_! \underline{k}_{Bun_{G,X}(C)} \in \mathcal{A}_G(C) = Shv(Bun_G(C))$$

X-period sheaf

Harmonic analysis on spherical varieties

Local picture [SV],[S],[SW]:

• Which representations appear in $L^2(X(K))$ determined by Arthur parameters factoring through $\check{G}_X \times SL_2 \to \check{G}$.

• Plancherel measure for spherical functions $L^2(X(K))^{G(O)}$ is Plancherel for G_X corrected by L-function of V_X:
$H_V, H_W \in Sph$ spherical Hecke operators $\Rightarrow$

$$\langle H_V * \Phi_X, H_W * \Phi_X \rangle = \int_{\check{A}_X^{cpt}/W_X} \chi_V(t)\overline{\chi}_W(t) \frac{det(Ad(1-t))}{det(1-F^{-1}t|_{V_X})} dt$$

Periods on spherical varieties

Global unramified picture [SV]:

• Which automorphic forms have nonvanishing X-periods (integral over $X \leftrightarrow$ pairing with $\Theta_X(\Phi)$) determined by Arthur parameters factoring through $\check{G}_X \times SL_2 \to \check{G}$.

Norm-squared of period given in terms of L-function of V_X:

$$\frac{|\Theta_X(\varphi)|^2}{\langle \varphi, \varphi \rangle} = \frac{L(\rho, V_X)}{L(\rho, Ad_{\check{G}_X})}$$

- Euler product version of X-Plancherel measure

The dual hyperspherical variety

[BZSV] Change of perspective: conjecture a duality operation on hyperspherical varieties

$$\boxed{G \circlearrowright M \Longleftrightarrow \check{G} \circlearrowright \check{M}}$$

Dual $\check{M}$ defined Tannakianly below. Conjecturally, it assembles all the dual data, as the Whittaker-twisted symplectic induction of S_X from $\check{G}_X$ to $\check{G}$:

$$\check{M} = T^*\check{G} \times^{\check{G}_X U}_{\mathfrak{g}_X^{\vee *} \oplus \mathfrak{u}_+} S_X$$

- e.g., $\check{M} = S_X$ in the strongly tempered case,
- $\check{M} = \check{G} \times^{\check{G}_X} V_X$ in tempered case.

Lifting

The duality highlights some symmetry between different periods, e.g.

- Tate and group cases self-dual
- Whittaker $\leftrightarrow$ pt,
- Gan-Gross-Prasad $\leftrightarrow$ Θ-correspondence.

Formally, duality implies that any period $G \circlearrowright M$ is a lift of a strongly tempered period $(\check{G}_X \circlearrowright S_X)^\vee$ for G_X

$\rightsquigarrow$ "explains" Knop H-C isomorphism, implies quantization of Knop's integration of invariant Hamiltonian flows à la [BZ-Gunningham].

Dual as categorified Plancherel measure

$\check{M}$ is geometrization of Plancherel measure for $G(O)\backslash X(K)$

Work in local geometric setting:

Plancherel

$$\langle H_V * \Phi_X, H_W * \Phi_X \rangle$$

lifts to

$$Hom_{Shv(G(O)\backslash X(K))}(\underline{H}_V * \underline{\Phi}_X, \underline{H}_W * \underline{\Phi}_X)$$

This data captured by internal endomorphism algebra

$$\mathcal{A}_X = End_{\underline{Sph}}(\Phi_X) \in Alg(\underline{Sph})$$

- an associative factorization algebra object; its cohomology is a [2-shifted] Poisson algebra.

Constructing the dual

Derived Geometric Satake[6]: $\underline{Sph} \xrightarrow{\simeq} Coh(\mathfrak{g}^{\vee *}[2])^{\check{G}}$

So Plancherel algebra $\mathcal{A}_X$ leads to affine Hamiltonian $G^\vee$-variety

$$\check{M} = Spec_{/\check{\mathfrak{g}}^*}(H^*(\mathcal{A}_X))$$

• $\check{M}//\check{G} = Spec(\mathcal{O}(\check{M})^G) \simeq \check{\mathfrak{c}}_X$ Poisson commutative, automatically hyperspherical!

Closely related to Coulomb branch construction[7] and electric-magnetic duality for boundary conditions[8] in $\mathcal{N} = 4$ SYM.

[6] Bezrukavnikov-Finkelberg
[7] Braverman-Finkelberg-Nakajima
[8] Gaiotto-Witten

The Local Geometric Conjecture

The Local Geometric Conjecture: There is an equivalence of categories

$$Shv(G(O)\backslash X(K)) \simeq QC^{[2]}(\check{M})^{\check{G}}$$

compatible with

- $\underline{Sph}$ actions
- Frobenius $\leftrightarrow \mathbb{G}_m$ action
- Poisson / factorization structure

The big picture

The local geometric conjecture is the basic building block for a meta-conjecture:

automorphic quantization of $G \circlearrowright M$ is Langlands dual to spectral quantization of $\check{G} \circlearrowright \check{M}$:

$$\boxed{\Theta_M \in \mathcal{A}_G \leftrightarrow \mathcal{L}_{\check{M}} \in \mathcal{B}_{\check{G}}}$$

Sea of Conjectures

	automorphic	spectral
global arithmetic: (numbers)	X-periods of automorphic forms	$\check{M}$-L-function of Galois reps
global geometric: (vector spaces)	X-periods of automorphic sheaves	$\check{M}$-L-sheaf
local arithmetic: (vector spaces)	spherical functions on $X(K)$	functions on $(\check{M})^{Frob}$
local geometric: (categories)	spherical sheaves on $X(K)$	quasicoherent sheaves on $\check{M}$

Structure

The different settings are related by strong compatibilities:

$$\textit{Geometric} \xrightarrow{\text{Trace of Frobenius}} \textit{Arithmetic}$$

$$\textit{Local} \xrightarrow{\text{Factorization homology / Euler product}} \textit{Global}$$

Can formulate as an equivalence of [morphisms of] algebraic field theories on curves[9] – algebraic model for $[1, 2, 3]$-dimensional part of a 4d TQFT with a boundary condition.

---

[9] Beilinson-Feigin-Mazur, Beilinson-Drinfeld

Spectral Quantization

Geometric quantization of 2-shifted symplectic varieties[10] $\check{M}$
- form of Rozansky-Witten 3d TQFT.

$\check{M}$ graded Hamiltonian $\check{G}$-variety $\rightsquigarrow \mathcal{L}_{\check{M}} \in \mathcal{B}_{\check{G}}$ invariants defined relative to spectral side of Langlands.

What does this mean?

• Global geometric setting: attach a vector space,
or relative version: sheaf $\mathcal{L}_{\check{M}}(C) \in QC^!(Loc_{\check{G}}(C))$

• Local geometric setting: attach a category. Unramified version: $QC(\check{M})$.

[10] Calaque-Pantev-Toën-Vezzosi-Vaquiè, Safronov

Spectral Quantization and Deformation Quantization

- $\mathcal{L}_{\check{M}}(\mathbb{P}^1)$: ring $\mathcal{O}(\check{M})$, with shifted Poisson bracket, deformation quantized to an associative factorization algebra[11] in the spherical Hecke category — matching the Plancherel algebra $\mathcal{A}_X \in \underline{Sph}$.

- $\mathcal{L}_{\check{M}}(\mathbb{P}^1)^{SO(2)}$: recover the *ordinary* (unshifted) deformation quantization of $\mathcal{O}(\check{M})$.

Polarized case: algebra $\mathcal{D}(\check{X})$, setting of Knop Harish-Chandra isomorphism

---

[11] In topology: [framed] E_3-algebra

Spectral Quantization and L-functions

Geometric home for L-functions:

$\check{G} \circlearrowright V$ representation $\rightsquigarrow$ L-function

$$\frac{1}{det(1 - t\rho(F))} = Tr_{gr}(F, Sym^{\bullet} V = \mathcal{O}(V^*))$$

Replace V^* by a $\check{G} \times \mathbb{G}_m$-variety $\check{X}$..

..or a graded Hamiltonian $\check{G}$-variety $\check{M}$ with spectral quantization!

Spectral Quantization and L-functions

Global version: L-sheaf $\mathcal{L}_{\check{X}}(C) = \check{\pi}_* \omega \in QC^!(Loc_{\check{G}}(C))$,

$$Loc_{\check{G},\check{X}}(C) = \{\check{G}\text{-loc. sys.} + \text{section of } \check{X}\text{-bundle}\} \xrightarrow{\check{\pi}} Loc_{\check{G}}(C)$$

categorifies sum of L-functions of local system / Galois representation over fixed points on $\check{X}$.

• $\check{M}$ not polarized: local-global compatibility defines "holonomic differential equation" for $\mathcal{L}_{\check{M}}(C)$.

Determined up to $\mathbb{Z}/2$-gerbe (at least away from poles of L-function)[12].

[12] work in progress

The Global Geometric Conjecture

The Global Geometric Conjecture:[13]

The geometric Langlands correspondence[14]

$$Shv(Bun_G(C)) \simeq QC^!(Loc_{\check{G}}(C))$$

intertwines the period sheaf[15] $\Theta_M(C)$ and the L-sheaf $\mathcal{L}_{\check{M}}(C)$.

[13] ignoring half-twists / normalizations

[14] de Rham, Betti or restricted

[15] after projection to nilpotent singular support