Harmonic Analysis and Gamma Functions on Symplectic Groups

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Preliminaries

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$$L(s, \pi, \rho) = \prod_{\mathfrak{p}} L(s, \pi_{\mathfrak{p}}, \rho);$$

- By Langlands, $L(s, \pi, \rho)$ (actually the partial L-function) is absolutely convergent for $\mathrm{Re}(s)$ large;

Preliminaries

Langlands' conjecture

$L(s, \pi, \rho)$ has a meromorphic continuation to $s \in \mathbb{C}$, and the functional equation

$$L(1-s, \pi^{\vee}, \rho) = \varepsilon(s, \pi, \rho) L(s, \pi, \rho)$$

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- The conjecture is known for a special list of (G,ρ);
- Methods: Godement-Jacquet (Tate), Rankin-Selberg; Langlands-Shahidi; Trace formula;

Preliminaries

Natural question

Establish the basic analytic properties for $L(s, \pi, \rho)$ through harmonic analysis on G (or related spherical varieties).

Godement-Jacquet

- R. Godement and H. Jacquet established the M.C. and F.E. of the standard L-function $L(s,\pi)$ of GL_n (over F-central simple algebras) via harmonic analysis on $\mathrm{GL}_n \hookrightarrow \mathrm{M}_n$, generalizing the work of Tate for $n = 1$ (when $n = 2$ it was also done in the last chapter of Jacquet-Langlands).

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- $G = \mathrm{GL}_n$;
- ${}^L G = \mathrm{GL}_n(\mathbb{C}) \times \mathcal{W}_F$, $\rho = \mathrm{Id} \otimes \{\text{trivial}\}$.

Godement-Jacquet: Local

For convenience, let $\mathfrak{p}$ be a non-archimedean place of F.

Ingredients

- Schwartz space $\mathcal{S}(G(F_{\mathfrak{p}})) = \mathcal{C}_c^\infty(\mathrm{M}_n(F_{\mathfrak{p}}))|_{G(F_{\mathfrak{p}})}$;

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Godement-Jacquet: Local theory

For $f \in \mathcal{S}(G(F_\mathfrak{p}))$, set

$$\mathcal{Z}(s, f, \varphi_{\pi_\mathfrak{p}}) = \int_{G(F_\mathfrak{p})} f(g)\varphi_{\pi_\mathfrak{p}}(g)|\det g|_{F_\mathfrak{p}}^{s+\frac{n-1}{2}} dg, \quad s \in \mathbb{C},$$

where $\varphi_{\pi_\mathfrak{p}} \in \mathcal{C}(\pi_\mathfrak{p})$ (the space of matrix coefficients of $\pi_\mathfrak{p}$).

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Theorem (Godement-Jacquet)

- *$\mathcal{Z}(s, f, \varphi_{\pi_\mathfrak{p}})$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large, and is a rational function in q^{-s};*

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- *there exists a rational function $\gamma(s, \pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ in q^{-s} such that the following functional equation holds for any $f \in \mathcal{S}(G(F_{\mathfrak{p}}))$*

$$\mathcal{Z}(1 - s, \mathcal{F}_{\psi_{\mathfrak{p}}}(f), \varphi^{\vee}_{\pi_{\mathfrak{p}}}) = \gamma(s, \pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})\mathcal{Z}(s, f, \varphi_{\pi_{\mathfrak{p}}}).$$

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- *Let* $1_\mathfrak{p}$ *be the characteristic function of* $\mathrm{M}_n(\mathfrak{o}_\mathfrak{p}) \subset \mathrm{M}_n(F_\mathfrak{p})$*. Then* $\mathcal{F}_{\psi_\mathfrak{p}}(1_\mathfrak{p}) = 1_\mathfrak{p}$ *and* $\mathcal{Z}(s, 1_\mathfrak{p}, \varphi_{\pi_\mathfrak{p}}) = L(s, \pi_\mathfrak{p})$ *for any unramified representation* $\pi_\mathfrak{p}$ *and* $\varphi_{\pi_\mathfrak{p}}$ *zonal spherical.*

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Ingredients

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- For $f \in \mathcal{S}(G(\mathbb{A}))$, consider

$$\mathcal{Z}(s, f, \varphi_{\pi}) = \int_{G(\mathbb{A})} f(g)\varphi_{\pi}(g)|\det g|_{\mathbb{A}}^{s+\frac{n-1}{2}} d^{\times} g, \quad s \in \mathbb{C},$$

where $\varphi_{\pi} \in \mathcal{C}(\pi)$.

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Theorem (Godement-Jacquet)

- *When* $\mathrm{Re}(s)$ *is sufficiently large,* $\mathcal{Z}(s, f, \varphi_\pi)$ *is absolutely convergent, and* $\mathcal{Z}(s, f, \varphi_\pi) = \prod_{\mathfrak{p}} \mathcal{Z}_{\mathfrak{p}}(s, f_{\mathfrak{p}}, \varphi_{\pi_{\mathfrak{p}}})$ *whenever* $f = \otimes_{\mathfrak{p}} f_{\mathfrak{p}}$ *is a pure tensor.*

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- $\mathcal{Z}(s, f, \varphi_\pi)$ *has a meromorphic continuation to* $s \in \mathbb{C}$*, and the functional equation*

$$\mathcal{Z}(1 - s, \mathcal{F}_\psi(f), \varphi_\pi^\vee) = \mathcal{Z}(s, f, \varphi_\pi)$$

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  holds.
- *Meromorphic continuation and functional equation follow from the* **Poisson summation formula** *for* $(\mathcal{S}(G(\mathbb{A})), \mathcal{F}_\psi)$.

Braverman-Kazhdan proposal

- Around 2000, A. Braverman and D. Kazhdan proposed a conjectural framework to establish the analytical properties of general automorphic L-functions $L(s, \pi, \rho)$.
- The prototype of the proposal is the theory of Godement and Jacquet.

For convenience, make the following additional assumptions (can be removed)

Assumptions

- G/F split;
- ρ is obtained from an irreducible injective representation of $G^\vee(\mathbb{C})$ with highest weight λ_ρ;
- $\sigma : G \to \mathbb{G}_m$ a character playing the role of det for GL_n;

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- For $f \in \mathcal{S}_\rho(G(F_\mathfrak{p}))$, set

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- For geometric reason, may set

$$n_\rho = \langle \rho_B, \lambda_\rho \rangle$$

  where ρ_B is the half sum of positive roots (Bouthier-Ngô-Sakellaridis).

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  where ρ_B is the half sum of positive roots (Bouthier-Ngô-Sakellaridis).
- In general different n_ρ differ by unramified shift;

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Schwartz space

- For any (G, ρ), there is an affine spherical embedding $G \hookrightarrow \mathcal{M}_\rho$, where $\mathcal{M}_\rho$ arises from the theory of reductive monoids studied by M. Putcha, L. Renner and E. Vinberg. It is expected that $\mathcal{S}_\rho(G(F_\mathfrak{p}))$ is connected with the geometry of $\mathcal{M}_\rho$;

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- There should exist $\mathbb{L}_{\rho,\mathfrak{p}} \in \mathcal{S}_\rho(G(F_\mathfrak{p}))^{K_\mathfrak{p} \times K_\mathfrak{p}}$ called the *basic function*, such that $\mathcal{Z}(s, \mathbb{L}_{\rho,\mathfrak{p}}, \varphi_{\pi_\mathfrak{p}}) = L(s, \pi_\mathfrak{p}, \rho)$ for any unramified representation $\pi_\mathfrak{p}$ and $\varphi_{\pi_\mathfrak{p}}$ zonal spherical;

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- For Godement-Jacquet, $\mathcal{M}_\rho = \mathrm{M}_n$, $\mathbb{L}_{\rho,\mathfrak{p}} = 1_\mathfrak{p}$.

Braverman-Kazhdan proposal: Local

Fourier transform

- For any $f \in \mathcal{C}_c^\infty(G(F_\mathfrak{p}))$,

$$\mathcal{F}_{\rho,\psi_\mathfrak{p}}(f)(g) = |\sigma(g)|^{-2n_\rho - 1}(\Phi_{\rho,\psi_\mathfrak{p}} * f^\vee)(g);$$

where $\Phi_{\rho,\psi_\mathfrak{p}}$ is an **invariant** distribution on $G(F_\mathfrak{p})$ such that

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- $\mathcal{F}_{\rho,\psi_\mathfrak{p}}$ extends to a unitary operator on $L^2(G(F_\mathfrak{p}), |\sigma(\cdot)|^{2n_\rho+1}dg)$ and $\mathcal{F}_{\rho,\psi_\mathfrak{p}} \circ \mathcal{F}_{\rho,\psi_\mathfrak{p}^{-1}} = \mathrm{Id}$;

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- For Godement-Jacquet, $\Phi_{\rho,\psi_\mathfrak{p}}(g) = \psi(\mathrm{tr}(g))|\det(g)|^n$.

Braverman-Kazhdan proposal: Local unramified

Theorem (L.)

- *For $\mathfrak{p}$ non-archimedean,*

$$\mathcal{S}_\rho(G(F_\mathfrak{p}))^{K_\mathfrak{p}\times K_\mathfrak{p}} = \mathbb{L}_{\rho,\mathfrak{p}} * \mathcal{C}_c^\infty(G(F_\mathfrak{p}))^{K_\mathfrak{p}\times K_\mathfrak{p}}$$

and

$$\Phi_{\rho,\psi_\mathfrak{p}}^{K_\mathfrak{p}} = \text{ Inverse Satake transform of } \gamma(-s-n_\rho, \pi_\mathfrak{p}, \rho^\vee, \psi_\mathfrak{p}).$$

The proposal is verified in full detail in unramified setting;

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The proposal is verified in full detail in unramified setting;

- *For* $\mathfrak{p}$ *archimedean, take* $\mathbb{L}_{\rho,\mathfrak{p}}$ *as the inverse Harish-Chandra transform of* $L(s, \pi_\mathfrak{p}, \rho)$*, then*

$$\mathbb{L}_{\rho,\mathfrak{p},s} = \mathbb{L}_{\rho,\mathfrak{p}}|\sigma(\cdot)|^s, \text{ and } \Phi_{\rho,\psi_\mathfrak{p},s}^{K_\mathfrak{p}} = \Phi_{\rho,\psi_\mathfrak{p}}^{K_\mathfrak{p}}|\sigma(\cdot)|^s$$

can be plugged into the Arthur-Selberg trace formula when $\mathrm{Re}(s)$ *large.*

Braverman-Kazhdan proposal: Global

Conjectural ingredients

- Schwartz space $\mathcal{S}_\rho(G(\mathbb{A})) = \bigotimes'_{\mathfrak{p}} \mathcal{S}_\rho(G(F_{\mathfrak{p}}))$ w.r.t. $\{\mathbb{L}_{\rho,\mathfrak{p}}\}_{\mathfrak{p}<\infty}$;

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- ρ-**Poisson summation formula** for $(\mathcal{S}_\rho(G(\mathbb{A})), \mathcal{F}_{\rho,\psi})$.

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- It is the first substantial case after the work of Godement-Jacquet;
- Establish the analytical theory of $L(s, \pi, \rho)$ following the approach of Godement-Jacquet, provide new evidence substantially for the Braverman-Kazhdan proposal.

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- The major work we need is the right normalization of the local intertwining operators appearing in doubling method;

The work of Jiang-Luo-Zhang

In the following, let F be a p-adic field.

- $G = \mathbb{G}_m \times \mathrm{Sp}_{2n}$;
- $\rho : G^\vee(\mathbb{C}) = \mathbb{C}^\times \times \mathrm{SO}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2n+1}(\mathbb{C})$;
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- The major work we need is the right normalization of the local intertwining operators appearing in doubling method;
- Piatetski-Shapiro and Rallis, Lapid-Rallis and other more recent works found the right normalization which gave the local Langlands γ-factor via doubling local zeta integrals.

Doubling method (Piatetski-Shapiro and Rallis)

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- $P = MN = \mathrm{Stab}(L_\Delta)$ a Siegel parabolic in Sp_{4n}, where $L_\Delta = \{(v, v) | \quad v \in F^{2n}\}$ is a Lagrangian;
- $$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n} \hookrightarrow \mathrm{Sp}_{4n} \to P \backslash \mathrm{Sp}_{4n}$$
  has Zariski open dense image, with stabilizer
  $$P \cap (\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}) = \mathrm{Sp}_{2n}^{\Delta} \hookrightarrow \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n};$$

The work of Jiang-Luo-Zhang

The following diagram illustrates the basic idea behind our work

$$\begin{array}{ccccc} & & \mathrm{Sp}_{4n} & & \\ & & \downarrow & & \\ M^{\mathrm{ab}}wN & \longrightarrow & X_P & \longleftarrow & M^{\mathrm{ab}}(\mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\}) \simeq \mathbb{G}_m \times \mathrm{Sp}_{2n} \end{array}$$

where $X_P = [P, P]\backslash\mathrm{Sp}_{4n}$, $w = (\mathrm{Id}_{2n}, -\mathrm{Id}_{2n}) \in \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$, $M^{\mathrm{ab}} = [M, M]\backslash M \simeq \mathbb{G}_m$.

- $wPw = P^-$, $M^{\mathrm{ab}}wN$ is Zariski open dense in X_P;
- $G = \mathbb{G}_m \times \mathrm{Sp}_{2n}$ is Zariski open dense in X_P;

Harmonic analysis on $M^{\mathrm{ab}}wN \hookrightarrow X_P$

Fourier transform

- For $f \in \mathcal{C}_c^\infty(X_P(F))$, define

$$\mathcal{F}_{X,\psi}(f)(g) := \int_{F^\times}^{\mathrm{pv}} \eta_{\mathrm{pvs},\psi}(x)|x|^{-\frac{2n+1}{2}} \int_{N(F)} f(wn\mathfrak{s}(x)g)dndx.$$

where $\mathfrak{s} : \mathbb{G}_m \to M$ is a section of $M \to [M, M]\backslash M \simeq \mathbb{G}_m$;

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where $\mathfrak{s} : \mathbb{G}_m \to M$ is a section of $M \to [M,M]\backslash M \simeq \mathbb{G}_m$;
- $\eta_{\mathrm{pvs},\psi}(x)$ is a distribution on $F^\times$, which is a key ingredient towards the understanding of $\mathcal{F}_{\rho,\psi}$ and $\mathcal{S}_\rho(G(F))$;

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  where $\mathfrak{s} : \mathbb{G}_m \to M$ is a section of $M \to [M, M]\backslash M \simeq \mathbb{G}_m$;
- $\eta_{\mathrm{pvs},\psi}(x)$ is a distribution on $F^\times$, which is a key ingredient towards the understanding of $\mathcal{F}_{\rho,\psi}$ and $\mathcal{S}_\rho(G(F))$;
- The definition of $\eta_{\mathrm{pvs},\psi}$ first appeared in [Braverman-Kazhdan, 2002], but that definition of $\eta_{\mathrm{pvs},\psi}$ did not carry enough analytical information for our work.

Abelian harmonic analysis

- To understand the analytical nature of $\eta_{\mathrm{pvs},\psi}$, we develop the local harmonic analysis associated to $\eta_{\mathrm{pvs},\psi}$ in the spirit of Braverman-Kazhdan proposal;

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- An explicit formula for $\eta_{\mathrm{pvs},\psi}$ is obtained from the functional equation associated to zeta integrals on the prehomogeneous space $(\mathrm{GL}_{2n+1}, S_{2n+1})$, where S_{2n+1} is the space of $(2n+1)\times(2n+1)$ symmetric matrices. More precisely, for a character χ, the following zeta integral is considered

$$\mathcal{Z}(s,f,\chi) = \int_{S_{2n+1}(F)} f(X)\chi(X)|\det X|^{s-(n+1)}dX;$$

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$$\mathcal{Z}(s,f,\chi)=\int_{S_{2n+1}(F)} f(X)\chi(X)|\det X|^{s-(n+1)}dX;$$

- The functional equation for the zeta integrals on (GL_m, S_m) is known by the work of Piatetski-Shapiro and Rallis, and T. Ikeda.

Abelian harmonic analysis

The following diagram illustrates the idea

$$\begin{array}{ccccccc}
\mathcal{C}_c^\infty(S_{2n+1}) & & & \xrightarrow{\text{Fourier transform}} & & & \mathcal{C}_c^\infty(S_{2n+1}) \\
\downarrow{\scriptstyle F.I.} & & & & & & \downarrow{\scriptstyle F.I.} \\
S^+_{n,\beta}(F^\times) & \xrightarrow[\simeq]{|\cdot|^{-2n}} & \mathcal{S}^+_{\mathrm{pvs}}(F^\times) & \xrightarrow{\mathfrak{L}=\mathfrak{L}_{\eta_{\mathrm{pvs}},\psi}} & \mathcal{S}^-_{\mathrm{pvs}}(F^\times) & \xleftarrow[\simeq]{|\cdot|^{n+1}} & S^-_{n,\beta}(F^\times)
\end{array}$$

where

- $F.I.$ is the fiber integration along $\det : S_{2n+1} \to F$;
- $\mathfrak{L}$ is the induced linear transform.

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Theorem (JLZ)

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  2. *for $|x| \ll 1$,*

$$f(x) = a_0^{+}(\mathrm{ac}(x))|x|^{-2n} + \sum_{i=0}^{n-1} a_{i,+}^{+}(\mathrm{ac}(x))|x|^{i-\frac{2n-1}{2}} + a_{i,-}^{+}(\mathrm{ac}(x))|x|^{i-\frac{2n-1}{2}}(-1)^{\mathrm{ord}(x)}$$

where a_0^{+} is a locally constant function on $\mathfrak{o}_F^{\times}$ that is $\mathfrak{o}_F^{\times}$-invariant, $a_{i,\pm}^{+}$ are locally constant functions on $\mathfrak{o}_F^{\times}$ that are $\mathfrak{o}_F^{\times 2}$-invariant, $\mathrm{ac}(x) = \frac{x}{|x|}$;

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- *In particular, $\mathcal{C}_c^{\infty}(F^{\times}) \hookrightarrow \mathcal{S}^{+}_{\mathrm{pvs}}(F^{\times})$ is of finite codimension;*

Paley-Wiener theorem for $\mathcal{S}^{\pm}_{\mathrm{pvs}}(F^\times)$

Theorem (JLZ)

- *Under Mellin transform* ($\int_{F^\times} f(x)\chi_s(x)dx$), $\mathcal{S}^{+}_{\mathrm{pvs}}(F^\times)$ *is captured by*

$$L(s,\chi)\prod_{i=0}^{n-1} L(2s+2i+1,\chi^2).$$

It follows from the description of G.C.D. for the zeta integral $\mathcal{Z}(s,\cdot,\chi)$ attached to (GL_m, S_m), which is established in our work (for χ unramified it is proved by Piatetski-Shapiro and Rallis).

Abelian harmonic analysis

Proposition (JLZ)

- *For any* $f \in \mathcal{S}^+_{\mathrm{pvs}}(F^\times)$, *there is the following functional equation after meromorphic continuation*

$$\int_{F^\times} \mathfrak{L}(f)\chi_{s+\frac{n+1}{2}}(t)^{-1}dt = \beta_\psi(\chi_s)\int_{F^\times} f(t)\chi_{s+\frac{2n+1}{2}}(t)dt$$

where

$$\beta_\psi(\chi_s) = \gamma(s - \frac{2n-1}{2}, \chi, \psi)\prod_{r=1}^{n}\gamma(2s-2n+2r, \chi^2, \psi).$$

Abelian harmonic analysis

Theorem (JLZ)

- *For* $k > 0$*, let* $\mathbf{1}_k$ *be the normalized characteristic function of* $1 + \varpi^k \mathfrak{o}_F$*, then*

$$\lim_{k\to\infty} \mathfrak{L}(\mathbf{1}_k)(x)$$

is stably convergent, i.e. for fixed $x \in F^\times$*, there exists* N *such that* $\mathfrak{L}(\mathbf{1}_k)(x) = \mathfrak{L}(\mathbf{1}_N)(x)$ *for any* $k > N$*;*

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- *Define*

$$\eta_{\mathrm{pvs},\psi}(x) = |x|^{-\frac{2n+1}{2}} \lim_{k\to\infty} \mathfrak{L}(\mathbf{1}_k)(x).$$

Then $\eta_{\mathrm{pvs},\psi}(x)$ *is locally constant on* $F^\times$.

Abelian harmonic analysis

Theorem (JLZ)

- *The generalized Fourier transform* $\mathfrak{L} = \mathfrak{L}_{\eta_{\mathrm{pvs}},\psi} : \mathcal{S}^{+}_{\mathrm{pvs}}(F^{\times}) \to \mathcal{S}^{-}_{\mathrm{pvs}}(F^{\times})$ *is given by the following principal value integral*

$$\mathcal{L}(f) = (\eta_{\mathrm{pvs},\psi}|\cdot|^{\frac{2n+1}{2}} * f^{\vee}), \quad f \in \mathcal{S}^{+}_{\mathrm{pvs}}(F^{\times}).$$

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- *For any character* $\chi_s = \chi|\cdot|^s$ *of* $F^\times$*, the following principal value integral is convergent whenever* $\mathrm{Re}(s)$ *is sufficiently small, and admits meromorphic continuation to* $s \in \mathbb{C}$*,*

$$\begin{aligned} \eta_{\mathrm{pvs},\psi}(\chi_s) &:= \eta_{\mathrm{pvs},\psi} * \chi_s(e) \\ &= \lim_{k\to\infty} \int^{\mathrm{pv}}_{q^{-k}\le|x|\le q^k} \eta_{\mathrm{pvs},\psi}(x)\chi_s(x^{-1})dx \\ &= \beta_\psi(\chi_s). \end{aligned}$$

Abelian harmonic analysis

- In conclusion, we develop a new type of harmonic analysis on $F^\times$ associated to $(\mathcal{S}^{\pm}_{\mathrm{pvs}}(F^\times), \mathfrak{L}_{\eta_{\mathrm{pvs},\psi}}, \beta_\psi(\chi_s))$.

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- It can be viewed as the abelian case of the Braverman-Kazhdan proposal.
- This abelian harmonic analysis plays the key role in our work.

Harmonic analysis on $M^{\mathrm{ab}}wN \hookrightarrow X_P$

Fix $f \in \mathcal{C}_c^\infty(X_P(F))$. Define

$$R_X(f)(g) := \int_{N(F)} f(wng)dn.$$

Proposition (JLZ)

- *The function in* $a \in F^\times$

$$F_g(a) := |a|^{(2n+1)} R_X(f)(\mathfrak{s}(a)g)$$

lies in $\mathcal{S}^+_{\mathrm{pvs}}(F^\times)$.

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- $\mathcal{L}_{\eta_{\mathrm{pvs},\psi}}(F_g)(a) = |a|^{2n+1}\mathcal{F}_{X,\psi}(f)(\mathfrak{s}^{-1}(a)g)$ *lies in* $\mathcal{S}^-_{\mathrm{pvs}}(F^\times)$.

Compatibility between $\mathcal{F}_{X,\psi}$ and the unnormalized intertwining operator $\mathrm{M}_w(s,\chi)$

Proposition (JLZ)

- *Let* $\mathcal{P}_{\chi_s} : \mathcal{C}_c^\infty(X_P(F)) \to \mathrm{I}(s,\chi) = \mathrm{Ind}_P^{\mathrm{Sp}_{4n}}(\chi_s)$,

$$\mathcal{P}_{\chi_s}(f)(g) = \int_{F^\times} \chi_s(a)|a|^{\frac{2n+1}{2}} f(\mathfrak{s}^{-1}(a)g)da.$$

Then $\mathcal{P}_{\chi_s^{-1}} \circ \mathcal{F}_{X,\psi}(f)(g)$ *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently small, and the following identity holds after meromorphic continuation*

$$\mathcal{P}_{\chi_s^{-1}} \circ \mathcal{F}_{X,\psi}(f)(g) = \beta_\psi(\chi_s)(\mathrm{M}_w(s,\chi) \circ \mathcal{P}_{\chi_s})(f)(g).$$

Basic properties of $\mathcal{F}_{X,\psi}$ and $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$

Define

$$\mathcal{S}_{\mathrm{pvs}}(X_P(F)) = \mathcal{C}_c^\infty(X_P(F)) + \mathcal{F}_{X,\psi}(\mathcal{C}_c^\infty(X_P(F))).$$

Proposition (JLZ)

- *$\mathcal{F}_{X,\psi}$ stabilizes $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$.*

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- *$\mathcal{F}_{X,\psi}$ stabilizes $\mathcal{S}_{\rm pvs}(X_P(F))$.*
- *$|2|^{n(2n+1)} \cdot \mathcal{F}_{X,\psi}$ extends to a unitary operator on $L^2(X_P(F))$ and $\mathcal{F}_{X,\psi} \circ \mathcal{F}_{X,\psi^{-1}} = |2|^{-2n(2n+1)}\mathrm{Id}$.*

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- *Via* $\mathcal{P}_{\chi_s}$, $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$ *projects onto the space of good sections* $\mathrm{I}^\dagger(s,\chi)$ *introduced by S. Yamana.*

Asymptotic of $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$

Proposition (JLZ)

A function $f \in \mathcal{C}^\infty(X_P(F))$ belongs to $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$ if and only if f is right $K_{\mathrm{Sp}_{4n}}$-finite, and as a function in $a \in F^\times$,

$$|a|^{2n+1} f(\mathfrak{s}_a^{-1} k)$$

belongs to $\mathcal{S}^-_{\mathrm{pvs}}(F^\times)$ for any fixed $k \in K_{\mathrm{Sp}_{4n}}$.

- Therefore functions in $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$ can be described by their asymptotic behavior near the singular point.

Asymptotic of $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$

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belongs to $\mathcal{S}_{\mathrm{pvs}}^-(F^\times)$ for any fixed $k \in K_{\mathrm{Sp}_{4n}}$.

- Therefore functions in $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$ can be described by their asymptotic behavior near the singular point.
- The support of functions in $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$ in $\overline{X}_P^{\mathrm{aff}}(F)$ is compact. In particular $\overline{X}_P^{\mathrm{aff}}(F) \backslash X_P(F) = \{\vec{0}\}$.

Harmonic analysis on $\mathbb{G}_m \times \mathrm{Sp}_{2n} \hookrightarrow X_P$

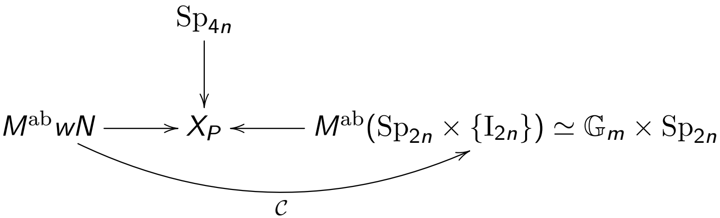
$$\begin{array}{ccc} & \mathrm{Sp}_{4n} & \\ & \downarrow & \\ M^{\mathrm{ab}} wN & \longrightarrow \; X_P \; \longleftarrow & M^{\mathrm{ab}}(\mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\}) \simeq \mathbb{G}_m \times \mathrm{Sp}_{2n} \end{array}$$

$$M^{\mathrm{ab}} wN \xrightarrow{\;\mathcal{C}\;} M^{\mathrm{ab}}(\mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\})$$

Proposition (JLZ)

- $\mathcal{C} : wN \to \mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\}$ *is given by the Cayley transform.*

Harmonic analysis on $\mathbb{G}_m \times \mathrm{Sp}_{2n} \hookrightarrow X_P$



Proposition (JLZ)

- *$\mathcal{C} : wN \to \mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\}$ is given by the Cayley transform.*
- *The Jacobian of $\mathcal{C}^{-1}$ is given by*

$$\mathrm{j}_{\mathcal{C}^{-1}}(h) = c_0 |\det(h - \mathrm{I}_{2n})|^{-(2n+1)}$$

where $c_0 = \frac{1}{\prod_{i=1}^{n} \zeta_F(2i)}$.

Harmonic analysis on $\mathbb{G}_m \times \mathrm{Sp}_{2n} \hookrightarrow X_P$

- For $f \in \mathcal{S}_{\mathrm{pvs}}(X_P(F))$, define

$$\phi_f(a,h) := f(\mathfrak{s}(a)^{-1} \cdot (h, \mathrm{I}_{2n}))|a|^{\frac{2n+1}{2}}.$$

Set

$$\mathcal{S}_\rho(G(F)) := \{\phi_f | \quad f \in \mathcal{S}_{\mathrm{pvs}}(X_P(F))\}.$$

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Set

$$\mathcal{S}_\rho(G(F)) := \{\phi_f| \quad f \in \mathcal{S}_{\mathrm{pvs}}(X_P(F))\}.$$

- Define

$$\Phi_{\rho,\psi}(a,h) := c_0 \cdot \eta_{\mathrm{pvs},\psi}(a\cdot \det(h+\mathrm{I}_{2n}))\cdot|\det(h+\mathrm{I}_{2n})|^{-\frac{2n+1}{2}}.$$

For $f \in \mathcal{C}_c^\infty(X_P(F))$, the ρ-Fourier transform is defined by

$$\mathcal{F}_{\rho,\psi}(\phi_f)(a,h) : = \int_{F^\times}^{\mathrm{pv}}\int_{\mathrm{Sp}_{2n}(F)} \Phi_{\rho,\psi}(ax,gh)\phi_f(x,g)dxdg.$$

Compatibility between $\mathcal{F}_{X,\psi}$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- For $f \in \mathcal{C}_c^\infty(X_P(F))$,

$$\phi_{\mathcal{F}_{X,\psi}(f)}(a,h) = |2|^{-n(2n+1)}\mathcal{F}_{\rho,\psi}(\phi_f)(2^{-2n}a, -h^{-1}).$$

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$$\phi_{\mathcal{F}_{X,\psi}(f)}(a,h) = |2|^{-n(2n+1)}\mathcal{F}_{\rho,\psi}(\phi_f)(2^{-2n}a, -h^{-1}).$$

- *In particular, we can extend the definition of* $\mathcal{F}_{\rho,\psi}$ *to* $\mathcal{S}_\rho(G(F))$ *via*

$$\phi_{\mathcal{F}_{X,\psi}(f)}(a,h) = |2|^{-n(2n+1)}\mathcal{F}_{\rho,\psi}(\phi_f)(2^{-2n}a, -h^{-1}).$$

Compatibility between $\mathcal{F}_{\rho,\psi}$ and the normalized intertwining operators $\mathrm{M}_w^\dagger(s,\chi,\psi)$

Proposition (JLZ)

For $h \in \mathrm{Sp}_{2n}(F)$ and $f \in \mathcal{S}_{\mathrm{pvs}}(X_P(F))$,

$$\mathcal{P}_{\chi_s^{-1}} \circ f_{\mathcal{F}_{\rho,\psi}(\phi_f)}((-h^{-1}, \mathrm{Id}_{2n}))$$

is well-defined for $\mathrm{Re}(s)$ sufficiently small, and the following identity holds after meromorphic continuation to $s \in \mathbb{C}$,

$$\mathrm{M}_w^\dagger(s,\chi,\psi) \circ \mathcal{P}_{\chi_s}(f)((h,\mathrm{I})) = \mathcal{P}_{\chi_s^{-1}} \circ f_{\mathcal{F}_{\rho,\psi}(\phi_f)}((-h^{-1},\mathrm{I})).$$

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Basic properties of $\mathcal{S}_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

Fix $\chi \otimes \pi \in \mathrm{Irr}(G(F))$. *Set*

$$\mathcal{Z}(s, f, \varphi) = \int_{F^\times \times \mathrm{Sp}_{2n}(F)} \phi(a, h)\varphi(a, h)|a|^{s-\frac{1}{2}} dadh,$$

with $\phi \in \mathcal{S}_\rho(G(F)), \varphi \in \mathcal{C}(\chi \otimes \pi)$.
The integral is absolutely convergent for $\mathrm{Re}(s)$ *large, and represents a rational function in* q^{-s}.

- It can be deduced from the asymptotic of functions in $\mathcal{S}_{\mathrm{pvs}}(X_P(F))$.

Basic properties of $\mathcal{S}_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- *After restriction, the linear functional $\mathcal{Z}(s,\cdot,\cdot)$ lies in*

$$\mathrm{Hom}_{G(F)\times G(F)}(\mathcal{C}_c^\infty(G(F))\otimes(\chi_{s-\frac{1}{2}}^{-1}\otimes\pi^\vee)\otimes(\chi_{s-\frac{1}{2}}\otimes\pi),\mathbb{C}),$$

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  where the latter space is of dimension 1*.*
- *By equivariant property there exists a rational function $\Gamma_{\rho,\psi}(s,\chi\otimes\pi)$ in q^{-s} such that*

$$\mathcal{Z}(1-s,\mathcal{F}_{\rho,\psi}(f),\varphi^\vee)=\Gamma_{\rho,\psi}(s,\chi\otimes\pi)\cdot\mathcal{Z}(s,f,\varphi).$$

Basic properties of $\mathcal{S}_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

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- *Let $\varphi_{\chi_s\otimes\pi} \in \mathcal{C}(\chi_s \otimes \pi)$. Then as distributions on $G(F)$, the following identity holds by meromorphic continuation,*

$$\mathcal{F}_{\rho,\psi}(\varphi^\vee_{\chi_s\otimes\pi}) = \Gamma_{\rho,\psi}(\frac{1}{2}, \chi_s \otimes \pi) \cdot \varphi_{\chi_s\otimes\pi}.$$

where for $f \in \mathcal{C}^\infty_c(G(F))$,

$$(\mathcal{F}_{\rho,\psi}(\varphi^\vee_{\chi_s\otimes\pi}), f)_G := (\varphi^\vee_{\chi_s\otimes\pi}, \mathcal{F}_{\rho,\psi}(f))_G$$

whenever the latter does not touch the poles.
In particular $\Gamma_{\rho,\psi}(s, \chi \otimes \pi)$ is a Gamma function in the sense of Gelfand and Graev.

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$$\Gamma_{\rho,\psi}(\frac{1}{2}, \chi_s \otimes \pi) \cdot \Gamma_{\rho,\psi^{-1}}(\frac{1}{2}, \chi_s^{-1} \otimes \pi^\vee) = 1.$$

Basic properties of $\Phi_{\rho,\psi}$

- Set $G_\ell = \{(a,h) \in G(F) = F^\times \times \mathrm{Sp}_{2n} | \quad |a| = q^{-\ell}\}$. Let ch_ℓ be the characteristic function of G_ℓ.
- Set $\Phi_{\rho,\psi,\ell} = \Phi_{\rho,\psi} \cdot \mathrm{ch}_\ell$.

Basic properties of $\Phi_{\rho,\psi}$

Theorem (JLZ)

- *The distribution* $\Phi_{\rho,\psi,\ell}$ *lies in the Bernstein center of* $G(F)$*. For* $\chi \otimes \pi \in \mathrm{Irr}(G(F))$*, set*

$$(\chi \otimes \pi)(\Phi_{\rho,\psi,\ell}) = f_\ell(\chi \otimes \pi)\mathrm{Id}_{\chi\otimes\pi}.$$

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- *The following identity holds after meromorphic continuation*

$$\sum_{\ell} f_\ell(\chi_s \otimes \pi) = \Gamma_{\rho,\psi}(\frac{1}{2}, \chi_s^{-1} \otimes \pi^\vee)$$

Verification

Corollary (JLZ)

- *Based on the work of Yamana, for any* $\chi \otimes \pi \in \mathrm{Irr}(G(F))$*, the following set*

$$\mathcal{I}_{\chi\otimes\pi} = \{\mathcal{Z}(s, \phi, \varphi) | \quad \phi \in \mathcal{S}_\rho(G(F)), \varphi \in \mathcal{C}(\chi \otimes \pi)\}$$

is a finitely generated fractional ideal of $\mathbb{C}[q^{-s}, q^s]$ *with generator* $L(s, \chi \otimes \pi, \rho)$*.*

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- *Based on the work of Lapid-Rallis, Ikeda and Kakuhama,* $\Gamma_{\rho,\psi}(s, \chi\otimes\pi) = \gamma(s, \chi\otimes\pi, \rho, \psi)$.

Thank you!