Twisted endoscopic character relation for Kaletha's regular supercuspidal L-packets

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December 2, 2020
(MIT Lie Groups Seminar)

What I am going to talk about

Main result (in progress)

Kaletha's local Langlands correspondence for toral regular supercuspidal representations satisfies the twisted endoscopic character relation.

Local Langlands correspondence

- Let F be a p-adic field.
- Let $\mathbf{G}$ be a connected reductive group over F ($G := \mathbf{G}(F)$).
- $\Pi(\mathbf{G}) := \{$irreducible admissible representations of $G\}/\sim$,
- $\Phi(\mathbf{G}) := \{L$-parameters of $\mathbf{G}\}/\sim$.

Local Langlands correspondence (LLC)

There exists a natural map with finite fibers:

$$\mathrm{LLC}_{\mathbf{G}} \colon \Pi(\mathbf{G}) \to \Phi(\mathbf{G}).$$

In other words, there exists a natural partition of the set $\Pi(\mathbf{G})$ into subsets (which are finite, called L-packets) parametrized by L-parameters:

$$\Pi(\mathbf{G}) = \bigsqcup_{\phi \in \Phi(\mathbf{G})} \Pi_{\phi}^{\mathbf{G}} \quad (\Pi_{\phi}^{\mathbf{G}} := \mathrm{LLC}_{\mathbf{G}}^{-1}(\phi)).$$

- LLC is still conjectural in general, but a number of results have been obtained.

Motivation: comparison of different constructions

Approach 1: Specialize the group. For example,

- GL_N; Harris–Taylor
- quasi-split classical groups (Sp_{2n}, SO_N, U_N); Arthur, Mok

Approach 2: Specialize the class of representations. For example,

- regular supercuspidal representations; Kaletha
  (works for tamely ramified groups)
  (he dropped the regularity recently; `arXiv:1912.03274`)

Q. Do the two approaches give the same LLC (on their "intersection")?

Theorem (O.-Tokimoto, 2019)

We assume that $p \neq 2$. Kaletha's and Harris–Taylor's LLC coincide for regular supercuspidal representations of GL_N.

Q. Kaletha's LLC $=$ Arthur's (Mok's) LLC?

⇝ To answer this problem, we must show that Kaletha's LLC satisfies the endoscopic character relation, which is the characterization of Arthur's LLC.

What is (twisted) endoscopy?

- Take a "twist" $\theta \curvearrowright \mathbf{G}$ ($\rightsquigarrow \hat{\theta} \curvearrowright \hat{\mathbf{G}}$).
- Roughly speaking, a (twisted) endoscopic group for $(\mathbf{G}, \theta)$ is a quasi-split connected reductive group $\mathbf{H}$ over F equipped with
  - $\hat{\xi}\colon {}^L\mathbf{H} \hookrightarrow {}^L\mathbf{G}$,
  - $s \in \hat{\mathbf{G}}$ such that $\hat{\xi}(\hat{\mathbf{H}}) = \mathrm{Cent}_{\hat{\mathbf{G}}}(s\hat{\theta})^\circ =$ (fixed points of $\mathrm{Int}(s) \circ \hat{\theta}$ in $\hat{\mathbf{G}})^\circ$.

- Take $(\mathbf{G}, \theta)$ to be GL_N and

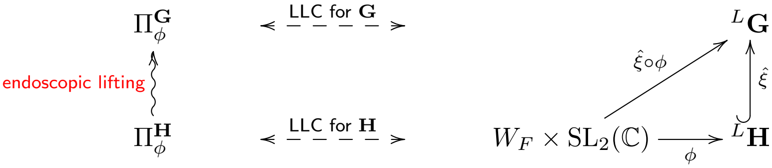
$$\theta(g) := J_N {}^t g^{-1} J_N^{-1}, \quad J_N := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{N-1} & & & \end{pmatrix}.$$

- Then a quasi-split Sp or SO is an endoscopic group for $(\mathbf{G}, \theta)$:

$\mathbf{H}$	SO_{2n+1}	Sp_{2n}	SO_{2n}
$\hat{\mathbf{H}}$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$
$\mathbf{G}$	GL_{2n}	GL_{2n+1}	GL_{2n}
$\hat{\mathbf{G}}$	$\mathrm{GL}_{2n}(\mathbb{C})$	$\mathrm{GL}_{2n+1}(\mathbb{C})$	$\mathrm{GL}_{2n}(\mathbb{C})$
s	1	1	$\mathrm{diag}(1, -1, 1, \ldots)$

What is endoscopic character relation?

The existences of $\mathrm{LLC}_{\mathbf{H}}$ and $\mathrm{LLC}_{\mathbf{G}}$ induces a lifting of L-packets.



Endoscopic character relation
= an identity between (twisted) characters of representations in $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$

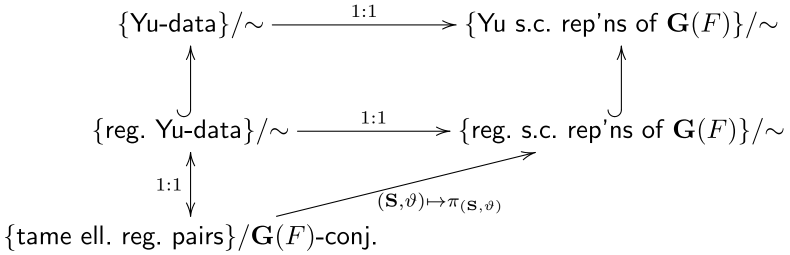
$\rightsquigarrow$ This characterizes the endoscopic lifting $\Pi_\phi^{\mathbf{H}} \mapsto \Pi_\phi^{\mathbf{G}}$ (**"Characters tell all"**).

Characterization of Arthur's LLC ($\mathbf{H}$ = classical, $\mathbf{G} = \mathrm{GL}_N$)

$\mathrm{LLC}_{\mathbf{H}}$ is the unique map such that $\Pi_\phi^{\mathbf{H}}$ and $\Pi_\phi^{\mathbf{G}}$ satisfy ECR.

Kaletha's reparametrization of Yu supercuspidals

- In the following, assume: $\mathbf{G}$ is tamely ramified and $p \gg 0$.
- In 2001, Yu constructed a certain wide class of supercuspidal representations.
- Kaletha reparametrized regular Yu-data by "tame elliptic regular pairs"; $(\mathbf{S}, \vartheta)$ consists of
  - a tamely ramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$ defined over F and
  - a "regular" character $\vartheta\colon \mathbf{S}(F) \to \mathbb{C}^{\times}$.



Construction of regular supercuspidal L-parameters

We take a regular supercuspidal L-packet datum $(\mathbf{S}, \hat{j}, \chi, \vartheta)$, which consists of

- a tamely ramified torus $\mathbf{S}$ defined over F,
- an embedding $\hat{j}\colon \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$ whose $\hat{\mathbf{G}}$-conjugacy class is Γ_F-stable,
- a set χ of χ-data for $(\mathbf{S}, \mathbf{G})$, and
- a "regular" character $\vartheta\colon \mathbf{S}(F) \to \mathbb{C}^\times$

satisfying several conditions.

Each $(\mathbf{S}, \hat{j}, \chi, \vartheta)$ produces an L-parameter ϕ and an L-packet $\Pi_\phi^{\mathbf{G}}$.

- LLC for $\mathbf{S}$ gives an L-parameter ϕ_ϑ of $\mathbf{S}$.
- By applying the Langlands–Shelstad construction to $(\hat{j}, \chi)$, we can extend $\hat{j}\colon \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$ to an L-embedding ${}^L j_\chi\colon {}^L\mathbf{S} \hookrightarrow {}^L\mathbf{G}$.
- Then we get an L-parameter ϕ of $\mathbf{G}$ (regular supercuspidal L-parameter):

$$\phi\colon W_F \xrightarrow{\phi_\vartheta} {}^L\mathbf{S} \xrightarrow{{}^L j_\chi} {}^L\mathbf{G}.$$

Construction of regular supercuspidal L-packets

- Put $\mathcal{J}^{\mathbf{G}}$ to be the $\mathbf{G}$-conjugacy class of embeddings $\mathbf{S} \hookrightarrow \mathbf{G}$ determined by $\hat{j}$ through the duality between $\mathbf{G}$ and $\hat{\mathbf{G}}$.
- Put
$$\mathcal{J}_G^{\mathbf{G}} := \{j \in \mathcal{J}^{\mathbf{G}} \mid \text{defined over } F\}/G\text{-conj.}$$
- For $j \in \mathcal{J}_G^{\mathbf{G}}$, define a tame elliptic regular pair $(\mathbf{S}_j, \vartheta_j)$ by $\mathbf{S}_j := j(\mathbf{S})$ and
$$\vartheta_j := (\vartheta \circ j^{-1}) \cdot (\epsilon \cdot \zeta)$$
Here $\epsilon \cdot \zeta \colon j(S) \to \mathbb{C}^\times$ is a character determined by root-theoretic information of $(\mathbf{S}_j, \mathbf{G})$ and χ.
- We define an L-packet of $\mathbf{G}$ by
$$\Pi_\phi^{\mathbf{G}} := \{\pi_j \mid j \in \mathcal{J}_G^{\mathbf{G}}\}, \quad \pi_j := \pi_{(\mathbf{S}_j, \vartheta_j)}.$$

Setting of the problem

- Let $\mathbf{G}$ be a quasi-split connected reductive group with θ.
- We follow the formalism of twisted space $(\mathbf{G}, \tilde{\mathbf{G}})$; $\tilde{\mathbf{G}} := \mathbf{G} \rtimes \theta$.
  ($\rightsquigarrow$ representation theory on $\tilde{G}$ = θ-twisted representation theory on G)
- Let $\mathbf{H}$ be an endoscopic group for $(\mathbf{G}, \theta)$.
- Suppose: $\mathbf{G}$ and $\mathbf{H}$ are tamely ramified and $p \gg 0$.
- Take a toral regular supercuspidal L-packet datum $(\mathbf{S}, \hat{j}, \chi, \vartheta)$ of $\mathbf{G}$.
  $\rightsquigarrow$ ϕ: associated toral regular supercuspidal L-parameter of $\mathbf{G}$
- Assume: ϕ factors through the L-group ${}^{L}\mathbf{H}$ of $\mathbf{H}$.
  $\rightsquigarrow$ ϕ is regarded as an L-parameter of $\mathbf{H}$ (again toral).

Proposition (θ-stable structure of $\Pi_{\phi}^{\mathbf{G}}$)

(1) $\Pi_{\phi}^{\mathbf{G}}$ is θ-stable, i.e., $\Pi_{\phi}^{\mathbf{G}} \circ \theta = \Pi_{\phi}^{\mathbf{G}}$ as sets.

(2) There is a canonical F-rational automorphism $\theta_{\mathbf{S}}$ of $\mathbf{S}$ such that, for any $j \in \mathcal{J}_{G}^{\mathbf{G}}$, we have $\pi_j \circ \theta \cong \pi_{j'}$, where $j' = \theta \circ j \circ \theta_{\mathbf{S}}^{-1}$.

- $\tilde{\mathcal{J}}_{G}^{\mathbf{G}} := \{ j \in \mathcal{J}_{G}^{\mathbf{G}} \mid j = \theta \circ j \circ \theta_{\mathbf{S}}^{-1} \}$ ($\rightsquigarrow$ parametrizes θ-stable members in $\Pi_{\phi}^{\mathbf{G}}$)

Precise statement of main result

Main result at present (ECR for $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$)

For an elliptic strongly regular s-s. element $\delta \in \tilde{G}$ having a norm in H, we have

$$\sum_{\pi \in \Pi_\phi^{\mathbf{G}}} \Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi)\tilde{\Theta}_\pi(\delta) = \sum_{\gamma \in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2} \Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta) \sum_{\pi_{\mathbf{H}} \in \Pi_\phi^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma),$$

where $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi) \in \mathbb{C}$ is a constant depending only on ϕ and π.

- $\Theta_{\pi_{\mathbf{H}}}$ is the (Harish-Chandra) character of $\pi_{\mathbf{H}} \in \Pi_\phi^{\mathbf{H}}$.
- $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi) \in \mathbb{C}$ is an explicit constant (spectral transfer factor). ($\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi) \neq 0$ if and only if π is θ-stable)
- $\tilde{\Theta}_\pi$ is the twisted character of π.
- γ runs over stable conjugacy classes of norms of δ in H.
- $D_{\mathbf{H}}$ and $D_{\tilde{\mathbf{G}}}$ denote the Weyl discriminants.
- $\Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta)$ is the transfer factor of Langlands–Shelstad–Kottwitz.

Toral regular supercuspidal representations

- Kaletha proved ECR for toral regular supercuspidal representations in the case of standard endoscopy:

standard endoscopy = twisted endoscopy with trivial θ

toral regular supercuspidal = arising from toral Yu-datum

- Roughly, a Yu-datum $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ consists of
  - a sequence $\vec{\mathbf{G}} = (\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G})$ of tame Levis,
  - a depth zero supercuspidal representation π_{-1} of $\mathbf{G}^0(F)$, and
  - a sequence $\vec{\phi} = (\phi_0, \ldots, \phi_d)$ of characters ϕ_i of $\mathbf{G}^i(F)$

  satisfying several technical conditions.
- A toral Yu-datum = Yu datum of the form

$$\big((\mathbf{S} \subsetneq \mathbf{G}), \mathbb{1}, (\vartheta, \mathbb{1})\big)$$

$\rightsquigarrow$ the tame elliptic regular pair corresponding to such a datum is given by $(\mathbf{S}, \vartheta)$, where $\vartheta = \phi_0$.

Standard endoscopic character relation

- Let $\mathbf{H}$ be an endoscopic group for $(\mathbf{G}, \mathrm{id})$.
- Take a toral L-packet datum $(\mathbf{S}, \hat{j}, \vartheta, \chi)$ of $\mathbf{G}$.
- Suppose: its L-parameter ϕ factors through ${}^L\mathbf{H} \hookrightarrow {}^L\mathbf{G}$.
  - ϕ is regarded as a toral L-parameter of $\mathbf{H}$.
  - get a toral L-packet datum $(\mathbf{S}_\mathbf{H}, \hat{j}_\mathbf{H}, \vartheta_\mathbf{H}, \chi_\mathbf{H})$ of $\mathbf{H}$.
- Recall that the members of $\Pi_\phi^\mathbf{G}$ and $\Pi_\phi^\mathbf{H}$ are parametrized by

$$\mathcal{J}_G^\mathbf{G} := \{j \in \mathcal{J}^\mathbf{G} \mid j\colon \mathbf{S} \hookrightarrow \mathbf{G}\text{: defined over } F\}/{\sim_G},$$

$$\mathcal{J}_H^\mathbf{H} := \{j_\mathbf{H} \in \mathcal{J}^\mathbf{H} \mid j_\mathbf{H}\colon \mathbf{S}_\mathbf{H} \hookrightarrow \mathbf{H}\text{: defined over } F\}/{\sim_H}.$$

Standard endoscopic character relation

$$\sum_{j \in \mathcal{J}_G^\mathbf{G}} \Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi_j)\Theta_{\pi_j}(\delta)$$

$$= \sum_{\gamma \in H/\mathrm{st}} \frac{D_\mathbf{H}(\gamma)^2}{D_\mathbf{G}(\delta)^2}\Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta) \sum_{j_\mathbf{H} \in \mathcal{J}_H^\mathbf{H}} \Theta_{\pi_{j_\mathbf{H}}}(\gamma),$$

Starting point: Adler–DeBacker–Spice character formula

- Let $(\mathbf{S}, \vartheta)$ be a toral tame elliptic regular pair of depth $r \in \mathbb{R}_{>0}$.
- Let $\delta \in G$ be an elliptic strongly regular semisimple element.
- Take a normal r-approximation $\delta = \delta_{<r} \cdot \delta_{\geq r}$; roughly speaking,
  - $\delta_{<r}$: "p-adically shallower than r"-part of δ (head),
  - $\delta_{\geq r}$: "p-adically deeper than r"-part of δ (tail).

("normal": $\delta_{\geq r}$ belongs to $\mathbf{G}_{\delta_{<r}} := \mathrm{Cent}_{\mathbf{G}}(\delta_{<r})^{\circ}$)

Prop (1st form of ADS formula)

$$\Theta_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ g\delta_{<r}g^{-1} \in S}} \Theta_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\bigl(\log(\delta_{\geq r})\bigr).$$

- $\pi_{(\mathbf{S},\vartheta)} = \text{c-Ind}_{K_{\sigma}}^{\mathbf{G}(F)} \sigma$ for certain open compact (mod-center) subgroup K_{σ}.
- $X^{*} \in (\mathrm{Lie}\,\mathbf{S})^{*}$ is an element representing $\vartheta|_{S_r}$.
- $\hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}$ is the Fourier transform of the orbital integral on $\mathrm{Lie}\,\mathbf{G}_{\delta_{<r}}$.
  (with respect to the $\mathbf{G}_{\delta_{<r}}(F)$-orbit of $g^{-1}X^{*}g$)

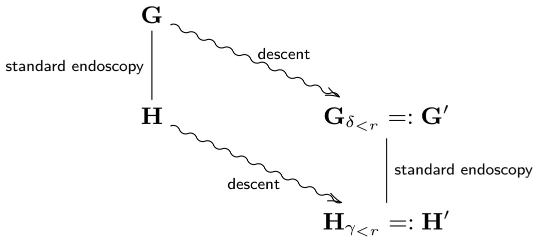
Langlands–Shelstad descent & reduction to Lie algebras

- For each $j \in \mathcal{J}_G^{\mathbf{G}}$ and $j_{\mathbf{H}} \in \mathcal{J}_H^{\mathbf{H}}$, we have

$$\mathbf{G}\text{-side:}\quad \Theta_{\pi_j}(\delta) = \sum \Theta_{\sigma_j}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X_j^* g}(\log(\delta_{\geq r}))$$

$$\mathbf{H}\text{-side:}\quad \Theta_{\pi_{j_{\mathbf{H}}}}(\gamma) = \sum \Theta_{\sigma_{\mathbf{H},j_{\mathbf{H}}}}(h\gamma_{<r}h^{-1}) \cdot \hat{\mu}^{\mathbf{H}_{\gamma_{<r}}}_{h^{-1}X^*_{\mathbf{H},j_{\mathbf{H}}}h}(\log(\gamma_{\geq r}))$$

The Langlands–Shelstad descent gives an endoscopic structure on $(\mathbf{H}_{\gamma_{<r}}, \mathbf{G}_{\delta_{<r}})$.



F.T. of orbital integrals = Lie algebra version of characters of representations

Waldspurger–Ngô transfer = Lie algebra version of SECR

Waldspurger–Ngô transfer on Lie algebras

Waldspurger–Ngô transfer on Lie algebras

$$\gamma(\mathfrak{g}') \sum_{X'^* \sim_{\mathrm{st}} X^*} \Delta_{\mathbf{H}',\mathbf{G}'}(Y^*, X'^*)\hat{\mu}^{\mathbf{G}'}_{X'^*}(X)$$

$$= \gamma(\mathfrak{h}') \sum_{Y/\mathrm{st}} \Delta_{\mathbf{H}',\mathbf{G}'}(Y, X) \sum_{Y'^* \sim_{\mathrm{st}} Y^*} \hat{\mu}^{\mathbf{H}'}_{Y'^*}(Y)$$

- index sets: G' (resp. H')-conj. classes in a $\mathbf{G}'$ (resp. $\mathbf{H}'$)-conj. class.

Recall: $\mathbf{G}$-side of SECR

$$\sum_{j \in \mathcal{J}^{\mathbf{G}}_G} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j)\Theta_{\pi_j}(\delta) = \sum_{j \in \mathcal{J}^{\mathbf{G}}_G} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) \sum_{\substack{g \in S_j \backslash G/G' \\ g\delta_{<r}g^{-1} \in S_j}} \Theta_{\sigma_j}({}^{g}\delta_{<r}) \cdot \hat{\mu}^{\mathbf{G}'}_{X^*_j, g}(\log(\delta_{\geq r}))$$

$\rightsquigarrow$ We need to consider a descent and transfer of index sets!

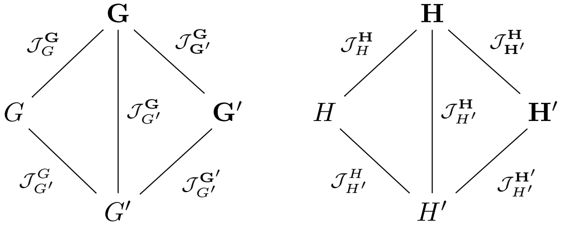
Transfer of index sets; Kaletha's descent lemma

- The index set of ADS formula is understood as

$$\mathcal{J}^{G}_{G'}(j) := \{k \in j \mid k\colon \mathbf{S} \hookrightarrow \mathbf{G}' \subset \mathbf{G}\text{: defined over } F\}/{\sim_{G'}}.$$

$$\{g \in S_j\backslash G/G' \mid g\delta_{<r}g^{-1} \in S_j\} \overset{1:1}{\longleftrightarrow} \mathcal{J}^{G}_{G'}(j)\colon g \mapsto \mathrm{Int}(g^{-1}) \circ j$$

- Combine $\mathcal{J}^{\mathbf{G}}_{G}$ with $\mathcal{J}^{G}_{G'}$ and divide it again via stable $\mathbf{G}'$-conjugacy.



- Kaletha's descent lemma relates $\mathcal{J}^{\mathbf{G}}_{\mathbf{G}'}$ to $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}'}$.
- The Waldspurger–Ngô transfer relates $\mathcal{J}^{\mathbf{G}'}_{G'}$ to $\mathcal{J}^{\mathbf{H}'}_{H'}$.

Computation of the contributions of r-heads

1st form of ADS formula

$$\Theta_{\pi_j}(\delta) = \sum_{\substack{g \in S_j \backslash G/G' \\ g\delta_{<r}g^{-1} \in S_j}} \Theta_\sigma(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}'}_{g^{-1}X_j^* g}\bigl(\log(\delta_{\geq r})\bigr).$$

- Adler–Spice ('09) computed Θ_σ at r-heads explicitly.
- DeBacker–Spice ('18) sophisticated it based on a root-theoretic language.
- Kaletha ('19) rewrote it via endoscopic invariants such as transfer factors.

ADS formula rewritten by Kaletha

$$\Theta_{\pi_j}(\delta) = \frac{e(\mathbf{G})}{e(\mathbf{G}')} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}} - \mathbf{T}_{\mathbf{G}'^*})}{D_{\mathbf{G}}(\delta)}$$

$$\cdot \sum_{\substack{g \in S_j \backslash G/G' \\ g\delta_{<r}g^{-1} \in S_j}} \Delta^{\mathbf{G},\mathrm{abs}}_{\mathrm{II}}(g\delta_{<r}g^{-1}) \cdot \vartheta \circ j^{-1}(g\delta_{<r}g^{-1}) \cdot \hat{\iota}^{\mathbf{G}'}_{g^{-1}X_j^* g}\bigl(\log(\delta_{\geq r})\bigr).$$

$\rightsquigarrow$ Finally, by putting $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) := \langle \mathrm{inv}(j_{\mathfrak{w}}, j), s\rangle$, we get SECR.

Twisted version of the Adler–DeBacker–Spice formula

- Let $\delta \in \tilde{G}$ be an elliptic strongly regular semisimple element.
  $\rightsquigarrow$ get a "twisted maximal torus" $(\mathbf{T}^{\diamondsuit}, \tilde{\mathbf{T}}^{\diamondsuit}) \subset (\mathbf{G}, \tilde{\mathbf{G}})$ defined over F.
  - $\tilde{\mathbf{T}}^{\diamondsuit}$ acts on $\mathbf{T}^{\diamondsuit}$ as $\theta_{\diamondsuit}$.
  - $\tilde{\mathbf{T}}^{\diamondsuit} = \mathbf{T}^{\diamondsuit}\delta$.
- Suppose: δ has a norm $\gamma \in H$.
  $\rightsquigarrow$ get a maximal torus $\mathbf{T}^{\flat} \subset \mathbf{H}$ containing γ.
- Since γ is a norm of δ, in fact we have $\mathbf{T}^{\diamondsuit}_{\theta_{\diamondsuit}} \cong \mathbf{T}^{\flat}$.
- The isogeny $\mathbf{T}^{\diamondsuit,\theta_{\diamondsuit}} \hookrightarrow \mathbf{T}^{\diamondsuit} \twoheadrightarrow \mathbf{T}_{\theta_{\diamondsuit}} \cong \mathbf{T}^{\flat}$ induces $T^{\diamondsuit,\theta_{\diamondsuit}}_{0+} \cong T^{\flat}_{0+}$.
  - Put $\delta_{\geq r} \in T^{\diamondsuit,\theta_{\diamondsuit}}_{0+}$ to be the image of $\gamma_{\geq r}$ via this isomoprhism.
  - Put $\delta_{<r} := \delta \cdot \delta_{\geq r}^{-1}$.

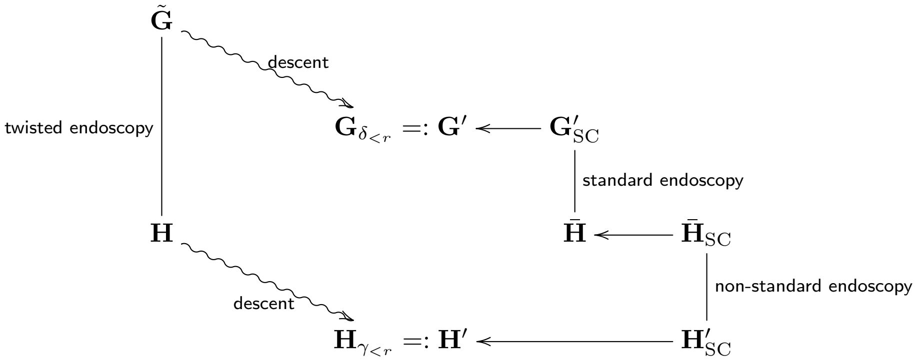
$\rightsquigarrow$ We get $\delta = \delta_{<r} \cdot \delta_{\geq r}$. (Note: $\delta_{<r} \in \tilde{G}$ and $\delta_{\geq r} \in G_{\delta_{<r}}$)

1st form of twisted ADS formula

$$\tilde{\Theta}_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in \tilde{S}}} \tilde{\Theta}_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\big(\exp^{-1}(\delta_{\geq r})\big).$$

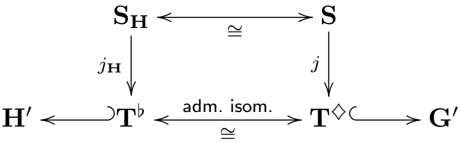
L'endoscopie tordue n'est pas si tordue

- Waldspurger constructed another connected reductive group $\bar{\mathbf{H}}$ relating $\mathbf{G}_{\delta_{<r}}$ to $\mathbf{H}_{\gamma_{<r}}$ via standard and non-standard endoscopy.
- Then he proved that
  - Fourier transforms of orbital integrals are transferred between $\mathbf{H}'_{\mathrm{SC}}$ and $\bar{\mathbf{H}}_{\mathrm{SC}}$,
  - transfer factor for $(\mathbf{H}, \tilde{\mathbf{G}})$ is descended to $(\bar{\mathbf{H}}, \mathbf{G}'_{\mathrm{SC}})$.

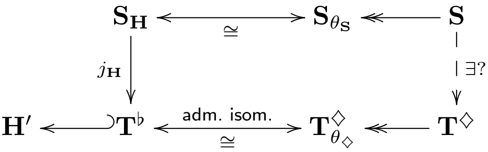


Twisted version of Kaletha's descent lemma

- In the standard case, a map $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}'} \to \mathcal{J}^{\mathbf{G}}_{\mathbf{G}'}$ of Kaletha's descent lemma was constructed via admissible isomorphisms:



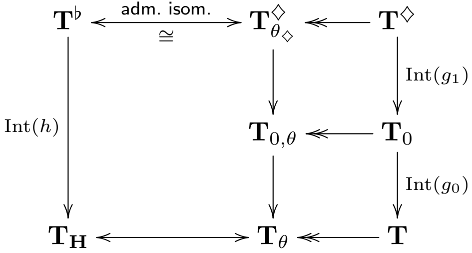
Unfortunately, we cannot simply immitate this construction in the twisted case.



$\rightsquigarrow$ We need to "rigidify" admissible isomorphisms in some way.

Waldspurger's "diagram"

- We utilize Waldspurger's notion of a diagram: $(\mathbf{T}^{\flat}, \mathbf{T}_0, \mathbf{T}^{\diamondsuit}, \mathbf{T}^{\natural}, h, g_0, g_1)$.



- A diagram encodes information about how an admissible isomorphism is given by conjugation (i.e., h, g_0, g_1).
- We can formulate a twisted version of Kaletha's descent lemma (which relates $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}'}$ to $\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}'}$) via diagrams.

$\rightsquigarrow$ get a comparison of index sets

Computation of the contributions of r-heads

1st form of twisted ADS formula

$$\tilde{\Theta}_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in \tilde{S}}} \tilde{\Theta}_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\bigl(\exp^{-1}(\delta_{\geq r})\bigr).$$

- The contribution of the head is eventually reduced to a computation of twisted characters of Weil representations of finite Heisenberg groups.
- We upgrade it to "twisted ADSK" by a case-by-case computation depending on the types of "restricted roots".

Proposition (twisted version of ADSK formula)

$$\tilde{\Theta}_{\pi_{j}}(\delta) = \frac{e(\mathbf{G}_{\theta})}{e(\mathbf{G}')} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}_{\theta}} - \mathbf{T}_{\mathbf{G}'^{*}})}{D_{\tilde{\mathbf{G}}}(\delta)}$$

$$\cdot \sum_{\substack{g \in S_{j}\backslash G/G' \\ g\delta_{<r}g^{-1} \in \tilde{S}_{\tilde{j}}}} \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}},\mathrm{abs}}(g\delta_{<r}g^{-1}) \cdot \tilde{\vartheta} \circ \tilde{j}^{-1}(g\delta_{<r}g^{-1}) \cdot \hat{\iota}^{\mathbf{G}'}_{g^{-1}X_{j}^{*}g}\bigl(\log(\delta_{\geq r})\bigr).$$

Spectral transfer factors

$$\sum_{\pi\in\Pi_\phi^{\mathbf{G}}} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi)\tilde{\Theta}_\pi(\delta) = \sum_{\gamma\in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2}\Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta) \sum_{\pi_{\mathbf{H}}\in\Pi_\phi^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma),$$

- We put

$$\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) := \Delta_{\mathrm{I,III}}\big(\gamma_{<r}\cdot\exp(X^*_{\mathbf{H},j_{\mathbf{H}}}),\delta_{<r}\cdot\exp(X^*_j)\big)\cdot\frac{\vartheta_{\mathbf{H}}\circ j_{\mathbf{H}}^{-1}(\gamma_{<r})}{\tilde{\vartheta}\circ\tilde{j}^{-1}(\delta_{<r})}.$$

$\rightsquigarrow$ In fact, $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi)$ depends only on π.

- standard case: $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi) = \langle\mathrm{inv}(j_{\mathfrak{w}}, j), s\rangle$ by Kaletha

Remark. I borrowed this idea from the argument of Mezo in his proof of the twisted ECR for L-packets of real reductive groups (constructed by Langlands).