Some aspects of parabolic induction for the general linear group over a p-adic field

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Based on joint works with Alberto Mínguez and Max Gurevich

Let F be a non-archimedean local field.
My goal is to review the status of what is known and conjectured about the question of irreducibility of parabolic induction

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n(F)} \pi_1 \otimes \pi_2$$

(normalized) parabolic induction from the standard parabolic subgroup of type (n_1, n_2).
There will be more questions than answers.
Please don't hesitate to interrupt me for any questions and comments, or simply to set the record straight!

Why GL_n ?

- Many aspects of representation theory of p-adic groups (e.g., L-packets, endoscopy) are simpler for GL_n. It is a benchmark (and a prerequisite) for understanding other groups (especially classical groups).
- Representation theory of GL_n was undertaken by Bernstein–Zelevinsky in the 1970s. They highlighted working with all n's together, i.e., considering

$$\oplus_{n\geq 0}\mathcal{R}(\mathrm{GL}_n(F)).$$

  It is a monoidal category, with parabolic induction as the tensor functor and transitivity of induction as the associativity constraints with the identity being the one-dimensional representation of $\mathrm{GL}_0 = 1$. It is a ring category (the tensor functor is bilinear and biexact).
- Connections between representation theory of $\mathrm{GL}_n(F)$ and quantum groups.

Consider a quiver Q of type A_n with the standard orientation

$$\bullet \to \bullet \to \cdots \to \bullet$$

A representation of Q is a collection of finite-dimensional $\mathbb{C}$-vector spaces $V_1, \dots, V_n$ together with linear transformations $T_i : V_i \to V_{i+1}$, $i = 1, \dots, n-1$; in other words a finite-dimensional graded vector space $V = \oplus_{i=1}^n V_i$ and a degree 1 (nilpotent) linear transformation $T : V \to V$.

This forms an abelian category. Its indecomposable objects are Jordan blocks (segments) $[i, j]$, indexed by $1 \le i \le j \le n$: $\dim V_k = 1$ if $k \in [i, j]$ and $V_k = 0$ otherwise; $T_k \ne 0$ iff $i \le k < j$.

The irreducible objects are $[i, i]$, $i = 1, \dots, n$.

Thus, the objects up to isomorphisms are indexed by multisegments, which are simply formal finite sums of segments.

This is of course a special case of Gabriel's theorem (1972) which classifies the indecomposable objects of a Dynkin diagram of a root system with an orientation by the positive roots – followed up by Bernstein–Gelfand–Ponomarev 1973.

Fix $V = \oplus_{i=1}^{n} V_i$ of graded dimension $\boldsymbol{d} = (d_1, \ldots, d_n)$ and set $V_i = d_i = 0$ if $i \notin \{1, \ldots, n\}$. Consider the vector space

$$E_{\rightarrow}(V) = \{T : V \rightarrow V | T(V_i) \subset V_{i+1} \text{ for all } i\}.$$

(This is the module variety (of dimension $\boldsymbol{d}$) of the path algebra of Q.) Then $\mathrm{GL}_{\boldsymbol{d}} = \prod_{i=1}^{n} \mathrm{GL}_{d_i}$ acts linearly on $E_{\rightarrow}(V)$ with finitely many orbits, indexed by multisegments of type $\boldsymbol{d}$.
Note that $\dim E_{\rightarrow}(V) = \sum_i d_i d_{i+1}$.
We could also consider the opposite quiver (reversing the arrows) and

$$E_{\leftarrow}(V) = \{T : V \rightarrow V | T(V_i) \subset V_{i-1} \text{ for all } i\}.$$

This is the dual space of $E_{\rightarrow}(V)$. Once again, $\mathrm{GL}_{\boldsymbol{d}}$-orbits in $E_{\leftarrow}(V)$ are indexed by multisegments of type $\boldsymbol{d}$.

Preprojective algebra and nilpotent varieties (Pyasetskiĭ 1975, Gelfand–Ponomarev 1979,...,Lusztig 1990-1)

Consider all orientations simultaneously, i.e., the "bipartisan" quiver $\bar{Q}$

$$\bullet \leftrightarrow \bullet \leftrightarrow \cdots \leftrightarrow \bullet$$

Fix V of graded dimension $\boldsymbol{d}$.

$$\Lambda_{\boldsymbol{d}} = \{(A, B) \in E_{\rightarrow}(V) \times E_{\leftarrow}(V) : AB = BA\}.$$

This is the module variety of the $\boldsymbol{d}$-dimensional modules of the finite-dimensional quotient of the path algebra of $\bar{Q}$ by the

$$\text{relations} \qquad \overrightarrow{e}_i \overleftarrow{e}_{i+1} - \overleftarrow{e}_i \overrightarrow{e}_{i-1}, \quad i = 1, \dots, n.$$

$\Lambda_{\boldsymbol{d}}$ is of pure dimension $\dim E_{\rightarrow}(V)$ and in fact a Lagrangian subvariety of $T^*(E_{\rightarrow}(V)) = E_{\rightarrow}(V) \times E_{\leftarrow}(V)$.
The irr. comp. of $\Lambda_{\boldsymbol{d}}$ are the closure of the conormal bundle of $\mathrm{GL}_{\boldsymbol{d}}$-orbits in $E_{\rightarrow}(V)$ (which are indexed by multisegments of type $\boldsymbol{d}$).

General results

Let C_i be irr. comp. of $\Lambda_{\boldsymbol{d}_i}$, $i = 1, 2$ and let $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$.
Denote by $C_1 \oplus C_2$ the $\mathrm{GL}_{\boldsymbol{d}}$-orbit of $\{x_1 \oplus x_2 : x_i \in C_i\}$.
This is an irreducible set.
We say that an irr. comp. C is generically indecomposable if the set $\{x \in C : x \text{ is indecomposable}\}$ has nonempty interior.

Theorem (Crawley-Boevey–Schröer (2002))

1. *(analogue of Krull-Remak-Schmidt) Any irr. comp. C of $\Lambda_{\boldsymbol{d}}$ can be written uniquely (up to permutation) as $\overline{C_1 \oplus \cdots \oplus C_k}$ where C_i are generically indecomposable.*
2. *Let C_1, C_2 be irr. comp.. Then $\overline{C_1 \oplus C_2}$ is an irr. comp. if and only if there exist $x_i \in C_i$, $i = 1, 2$ such that* $\mathrm{Ext}^1(x_1, x_2) = \mathrm{Ext}^1(x_2, x_1) = 0$. *(This is a open condition in $(x_1, x_2) \in \Lambda_{\boldsymbol{d}_1} \times \Lambda_{\boldsymbol{d}_2}$.)*

Remarks

- The results above hold for the module variety of the d-dimensional modules of an arbitrary finite-dimensional ring. (Or more generally, the $\boldsymbol{d}$-dimensional modules of a finite-dimensional ring with orthogonal idempotents $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$.)
- By Voigt's lemma (1974), for any $x \in \Lambda_{\boldsymbol{d}}$ with $\mathrm{GL}_{\boldsymbol{d}}$-orbit $\mathcal{O}(x)$

$$\mathrm{Ext}^1(x, x) \simeq N_x(\mathcal{O}(x)) = T_x\Lambda_{\boldsymbol{d}} / T_x\mathcal{O}(x)$$

  where $T_x\Lambda_{\boldsymbol{d}}$ is the tangent space of the scheme $\Lambda_{\boldsymbol{d}}$ at x.
- In the preprojective case, $\mathrm{Ext}^1(x, x')$ and $\mathrm{Ext}^1(x', x)$ are in duality (and in particular, have the same dimension) for any $x \in \Lambda_{\boldsymbol{d}}$ and $x' \in \Lambda_{\boldsymbol{d}'}$. Moreover,

$$\operatorname{codim} \mathcal{O}(x) = \dim \Lambda_{\boldsymbol{d}} - \dim \mathcal{O}(x) = \dim T_x\Lambda_{\boldsymbol{d}} - \dim \Lambda_{\boldsymbol{d}}.$$

  It follows that $\dim \mathrm{Ext}^1(x, x) = 2 \operatorname{codim} \mathcal{O}(x)$ and therefore

$$\dim \mathrm{Ext}^1(x, x') = \operatorname{codim} \mathcal{O}(x \oplus x') - \operatorname{codim} \mathcal{O}(x) - \operatorname{codim} \mathcal{O}(x').$$

Rigid modules

We say that $x \in \Lambda_{\boldsymbol{d}}$ is rigid if the following equivalent conditions are satisfied.

1. $\mathrm{Ext}^1(x, x) = 0$.
2. $\mathcal{O}(x)$ is open in $\Lambda_{\boldsymbol{d}}$.
3. $\mathcal{O}(x)$ is an open subscheme of $\Lambda_{\boldsymbol{d}}$.
4. The Zariski closure $\overline{\mathcal{O}(x)}$ is an irr. comp. of $\Lambda_{\boldsymbol{d}}$.
5. $\dim \mathrm{End}(x) = \dim \mathrm{GL}_{\boldsymbol{d}} - \dim \Lambda_{\boldsymbol{d}}$.
6. $\dim \mathrm{End}(x) \leq \dim \mathrm{GL}_{\boldsymbol{d}} - \dim \Lambda_{\boldsymbol{d}}$.
7. The scheme $\Lambda_{\boldsymbol{d}}$ is smooth at x.

This condition can be checked by linear algebra.
If $x_1 \in \Lambda_{\boldsymbol{d}_1}$ and $x_2 \in \Lambda_{\boldsymbol{d}_2}$ are rigid, then

$$x_1 \oplus x_2 \text{ is rigid } \iff \mathrm{Ext}^1(x_1, x_2) = 0 \iff \mathrm{Ext}^1(x_2, x_1) = 0.$$

Rigid irr. comp.

An irr. comp. C of $\Lambda_{\boldsymbol{d}}$ is called rigid if it satisfies the following equivalent conditions.

1. C contains a rigid module.
2. C contains a (unique) open $\mathrm{GL}_{\boldsymbol{d}}$-orbit.
3. The scheme $\Lambda_{\boldsymbol{d}}$ is generically reduced at C.

In this case, the open orbit in C consists of the rigid modules in C; it is contained in the conormal bundle whose closure in C.

$$\text{rigid irr. comp.} \longleftrightarrow \text{rigid modules}/\,\mathrm{GL}_{\boldsymbol{d}}$$

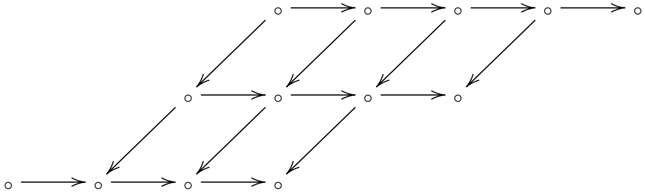
The role of rigid modules and irr. comp. was highlighted in the work of Geiss–Leclerc–Schröer (early 2000s –).
The rigidity condition for an irr. comp. can be checked probabilistically by linear algebra.

Question

Is there a simple combinatorial criterion for the rigidity of an irr. comp., or at least a deterministic algorithm?

Examples

- Suppose that C is the irr. comp. corresponding to a multisegment $\sum_{i=1}^{r}[a_i, b_i]$ such that $a_1 \le \cdots \le a_r$ and $b_1 \ge \cdots \ge b_r$. (Any two segments are comparable by inclusion.) Then C is rigid. In fact, in this case $C = E_{\to}(V)$.
- Similarly if $b_i = a_i$ for all i (all segments are singletons). In this case $C = E_{\leftarrow}(V)$.
- Assume $[a_i, b_i] = [i, n - r + i]$, $i = 1, \ldots, r$. We get the proj. indecomp. module p_r corresponding to the r-th simple root.
- More generally suppose that $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_r$. (We call such C special.) Then C is rigid.



A non-rigid example (Geiss–Schröer 2005, following Leclerc 2003)

For $n \leq 4$ all irr. comp. are rigid. ($\Lambda_{\boldsymbol{d}}$ is representation-finite.)
Consider $n = 5$, $\boldsymbol{d} = (1, 2, 2, 2, 1)$ ($\dim \Lambda_{\boldsymbol{d}} = 12$, $\dim \mathrm{GL}_{\boldsymbol{d}} = 14$) and the irr. comp. C with multisegment

$$[4, 5] + [2, 4] + [3, 3] + [1, 2].$$

C is the closure of a one-parameter family of 11-dimensional orbits and C is indecomposable. If $\mathcal{O}(x) \neq \mathcal{O}(y)$ then $\dim \mathrm{Hom}(x, y) = 2$ and $\mathrm{Ext}^1(x, y) = 0$, but $\dim \mathrm{End}(x) = 3$ and $\dim \mathrm{Ext}^1(x, x) = 2$.
Thus, $\overline{C \oplus C}$ is an irr. comp. even though there is a short exact sequence

$$0 \to x \to p_2 \oplus p_4 \to x \to 0$$

where as before p_2 and p_4 have multisegments

$$[1, 4] + [2, 5] \text{ and } [1, 2] + [2, 3] + [3, 4] + [4, 5].$$

Relation to representation theory

By Zelevinsky's classification (1980), there is a bijection

$$C \to \pi_C$$

between the irr. comp. of $\Lambda_{\boldsymbol{d}}$ (i.e., multisegments of type $\boldsymbol{d}$) and the irreducible subquotients (up to isomorphism) of

$$\overbrace{|\cdot| \times \cdots \times |\cdot|}^{d_1} \times \cdots \times \overbrace{|\cdot|^n \times \ldots |\cdot|^n}^{d_n}$$

(a representation of $\mathrm{GL}_{d_1+\cdots+d_n}(F)$).
Also, Lusztig's canonical bases (1990) of $U(\mathfrak{sl}_{n+1})^{\boldsymbol{d}}$ (the $\boldsymbol{d}$-graded piece of the positive part of the universal enveloping algebra of type A_n) are indexed by irr. comp. of $\Lambda_{\boldsymbol{d}}$.
Dually, if N is the maximal unipotent subgroup of GL_{n+1}, then $\mathbb{C}[N]$ is isomorphic to the subring of the Bernstein–Zelevinsky ring of representations of $\mathrm{GL}_k(F)$, $k \geq 0$ generated by $|\cdot|, \ldots, |\cdot|^n$. The dual canonical basis corresponds to the irreducible representations (Ariki, Grojnowski, Leclerc, Nazarov, Thibon, Zelevinsky)

Going back to the previous example
if C, C_1, C_2 corresponds to

$$\mathfrak{m} = [4,5] + [2,4] + [3,3] + [1,2]$$

$$\mathfrak{m}_1 = [1,4] + [2,5], \quad \mathfrak{m}_2 = [1,2] + [2,3] + [3,4] + [4,5],$$

then (Leclerc, 2003)

$$\pi_C \times \pi_C = \pi_{\overline{C \oplus C}} + \pi_{C_1} \times \pi_{C_2} = \pi_{\overline{C \oplus C}} + \pi_{\overline{C_1 \oplus C_2}}.$$

$\overline{C \oplus C}$ and $\overline{C_1 \oplus C_2}$ have multisegments $\mathfrak{m} + \mathfrak{m}$ and $\mathfrak{m}_1 + \mathfrak{m}_2$.

Conjecture 1 (Geiss–Schröer 2005 , after Marsh–Reineke)

Let C_i be irr. comp. of $\Lambda_{\mathbf{d}_i}$, $i = 1, 2$. Assume that

$$\text{there exist nonempty open subset } U_i \subset C_i \text{ such that} \quad \text{Ext}^1(x_1, x_2) = 0 \text{ for all } x_i \in U_i,\ i = 1, 2. \tag{*}$$

Then $\pi_{C_1} \times \pi_{C_2}$ is irreducible.

- As far as I know, the conjecture is wide open in general.
- Strong form: the converse also holds.
- If $C_1 = C_2$, the condition (*) is that C_1 is rigid.
- In general, (*) implies that $\overline{C_1 \oplus C_2}$ is an irr. comp..
- The converse holds if C_1 (say) is rigid, in which case the condition (*) is that $\text{Ext}^1(x_1, x_2) = 0$ for a rigid $x_1 \in C_1$ and generic $x_2 \in C_2$. This condition can be checked efficiently by a probabilistic algorithm.
- If neither C_i is rigid (and $C_1 \neq C_2$) then it is unclear how to check (*) algorithmically.

Special case: type A_{2n-1}, $\boldsymbol{d} = (1, 2, \ldots, n, n-1, \ldots, 1)$ $(\sum d_i = n^2)$

$$\overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \to \overset{n}{\bullet} \to \overset{n-1}{\bullet} \to \cdots \to \overset{1}{\bullet}$$

Consider the following open, $\mathrm{GL}_{\boldsymbol{d}}$-invariant subset of $E_{\to}(V)$

$$E^{\flat}_{\to}(V) = \{T \in E_{\to}(V) : T\big|_{V_i} \text{ is injective } \forall i < n \text{ and surjective } \forall i \geq n\}$$

Let X be the (complete) flag variety of GL_n.
The map $E^{\flat}_{\to}(V) \to X \times X$ given by $T \mapsto (\mathcal{F}_1(T), \mathcal{F}_2(T))$ where

$$\mathcal{F}_1(T) : \; 0 \subsetneq T^{n-1}(V_1) \subsetneq T^{n-2}(V_2) \subsetneq \cdots \subsetneq T(V_{n-1}) \subsetneq V_n,$$

$$\mathcal{F}_2(T) : \; 0 \subsetneq \mathrm{Ker}(T\big|_{V_n}) \subsetneq \mathrm{Ker}(T^2\big|_{V_n}) \subsetneq \cdots \subsetneq \mathrm{Ker}(T^{n-1}\big|_{V_n}) \subsetneq V_n,$$

is a principal $\prod_{i \neq n} \mathrm{GL}(V_i)$-bundle. Hence, we get an isomorphism of GL_n-varieties (cf. Kashiwara–Saito 1997)

$$E^{\flat}_{\to}(V) / \prod_{i \neq n} \mathrm{GL}(V_i) \longleftrightarrow X \times X.$$

Thus, the $\mathrm{GL}_{\mathbf{d}}$-orbits in $E^{\flat}_{\rightarrow}(V)$ correspond to the GL_n-orbits in $X \times X$, which are parameterized by the symmetric group S_n.
If Y_w, $w \in S_n$ is a GL_n-orbit in $X \times X$ (Bruhat cell), then the corresponding irr. comp. C_w of $\Lambda_{\mathbf{d}}$ has multisegment

$$[1, w(1) + n - 1] + \cdots + [n, w(n) + n - 1].$$

Denote by X_w the closure of Y_w (Schubert variety).
For example, $X_e = Y_e = \Delta X$ (diagonal), Y_{w_0} open, $X_{w_0} = X \times X$.

Theorem (•+Mínguez, 2018)

The following conditions on $w \in S_n$ are equivalent.

1. *C_w is rigid.*
2. *The conormal bundle of $Y_w \subset X$ has an open GL_n-orbit.*
3. *$X_{w_0 w}$ is (rationally) smooth.*
4. *$\pi_{C_w} \times \pi_{C_w}$ is irreducible.*
5. *(Lakshmibai–Sandhya, 1990) w is 1324 and 2143 avoiding.*

The case $w = 1324$ is essentially Leclerc's example.

Remarks

Conditions 2 and 3 are purely geometric.
Their equivalence leads to the following

Conjecture 2 (Mellit)

Let $x, w \in S_n$ with $Y_w \subset X_x$ (i.e., $w \leq x$). Suppose that X_x is smooth. Then the following conditions are equivalent

1. *The conormal bundle of $Y_w \subset X_x$ has an open GL_n-orbit.*
2. *The smooth locus of $X_{w_0 w}$ contains $Y_{w_0 x}$.*

We proved this conjecture (along with a representation-theoretic criterion) for x 231 avoiding (which implies that X_x is smooth).
The current proof is not conceptual.
In general, one can realize in a similar way the GL_n-action on $P \backslash \mathrm{GL}_n \times Q \backslash \mathrm{GL}_n$ for any parabolic subgroups P and Q of GL_n.
Unfortunately, the naive analogue of the theorem in this context is not true – nor do we have a conjectural replacement for the smoothness condition.

Theorem (translation of Kang–Kashiwara–Kim–Oh (2015))

The following conditions are equivalent for a rep'n π *of* $\mathrm{GL}_n(F)$.

1. $\pi \times \pi$ *is irreducible.*
2. $\mathrm{End}_{\mathrm{GL}_{2n}(F)}(\pi \otimes \pi) = \mathbb{C}$.
3. *The normalized intert. oper.* $\pi \times \pi \to \pi \times \pi$ *is a scalar.*

Under these conditions, for any irreducible representation σ *of* $\mathrm{GL}_m(F)$ *the socle of* $\pi \times \sigma$ *is irreducible and occurs with mult. one in* $\mathrm{JH}(\pi \times \sigma)$. *It is the image of the intert. oper.* $\sigma \times \pi \to \pi \times \sigma$.

This result gives an interesting perspective on Bernstein's result (1983) on the irreducibility of parabolic induction of unitarizable representations (proved by a completely different method). It yields a simplification of the proof of Tadic's classification of the unitary dual of $\mathrm{GL}_n(F)$ (1986).
Recall that conjecturally $\pi \times \pi$ is irreducible if and only if the corresponding irr. comp. of $\Lambda_{\boldsymbol{d}}$ is rigid.

Subrepresentations

Let C_i be an irr. comp. of $\Lambda_{\boldsymbol{d}_i}$, $i = 1, 2$ and $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$. Let

$$S = \{(x_1, x_2) \in C_1 \times C_2 : \dim \operatorname{Ext}^1(x_1, x_2) \text{ is minimal}\},$$

an open subset of $C_1 \times C_2$. The $\mathrm{GL}_{\boldsymbol{d}}$-invariant set

$$\mathcal{E}(C_1, C_2) = \{x \in \Lambda_{\boldsymbol{d}} : \exists \text{ a short exact sequence } 0 \to x_2 \to x \to x_1 \to 0 \text{ with } (x_1, x_2) \in S\}$$

is irreducible (Crawley-Boevey–Schröer, 2002). Moreover, $C = \overline{\mathcal{E}(C_1, C_2)}$ is an irr. comp. (Rami Aizenbud)

Conjecture 3

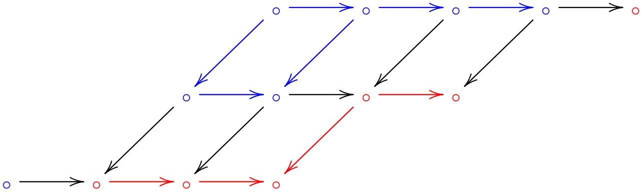
π_C is a subrepresentation of $\pi_{C_1} \times \pi_{C_2}$.

If true, a generic extension of a generic $x_1 \in C_1$ by a generic $x_2 \in C_2$ determines an irreducible subrepresentation of $\pi_{C_1} \times \pi_{C_2}$. (It is easy to compute $\overline{\mathcal{E}(C_1, C_2)}$ by a probabilistic algorithm.)

Example

The following diagram is an extension of the red part by the blue part.

Recall that the dots represent a basis for V, the grading is by the horizontal position; the horizontal arrows define $A \in E_{\rightarrow}(V)$ and the diagonal arrows define $B \in E_{\leftarrow}(V)$.



Corroboration

Theorem (•+Mínguez, 2016, 2020)

Suppose that C_1 or C_2 is a direct sum of special irr. comp.. Then there is a simple combinatorial way to determine the multisegment of $C = \overline{\mathcal{E}(C_1, C_2)}$ from the multisegments $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of C_1 and C_2. Moreover, $\pi_C = \mathrm{soc}(\pi_{C_1} \times \pi_{C_2})$.

A special case

Let $\mathfrak{m}_i$ be the multisegments of C_i and write $\mathfrak{m}_i = \sum_{j \in I_i} \Delta_j$, $i = 1, 2$ with $I_1 \cap I_2 = \emptyset$.
Define a bipartite graph $\mathcal{G} = (X, Y, E)$ by

$$\begin{aligned} X =& \{(r,s) \in (I_1 \times I_2) \cup (I_2 \times I_1) : \Delta_r \prec \Delta_s\} \\ Y =& \{(r,s) \in (I_1 \times I_2) \cup (I_2 \times I_1) : \overset{\rightarrow}{\Delta}_r \prec \Delta_s\} \\ E =& \{((r,s),(r,t)) \in X \times Y : \overset{\rightarrow}{\Delta}_s \prec \Delta_t\} \cup \\ & \{((r,t),(s,t)) \in X \times Y : \overset{\rightarrow}{\Delta}_s \prec \Delta_r\} \end{aligned}$$

where for $\Delta = [a, b]$, $\Delta' = [a', b']$ we write

$$\Delta \prec \Delta' \iff a \leq a' \leq b \leq b', \quad \overset{\rightarrow}{\Delta} = [a+1, b+1].$$

Theorem

If C_1 or C_2 is special then $\pi_{C_1} \times \pi_{C_2}$ is irreducible if and only if there exists a matching in $\mathcal{G}$ which covers all vertices of Y.

Odds and ends

Question

Is there a relation between the set of irreducible subquotients of $\pi_{C_1} \times \pi_{C_2}$ and the set of irr. comp. containing $C_1 \oplus C_2$?

For instance, it is clear that if C_i correspond to $\mathfrak{m}_i$, $i = 1, 2$ and C corresponds to $\mathfrak{m}_1 + \mathfrak{m}_2$ then $C \supset C_1 \oplus C_2$.

Question

Is there a practical way to check whether a given $x \in \Lambda_{\boldsymbol{d}}$ is contained in a given irr. comp.?

Standard modules and Robinson–Schensted–Knuth

Let C be an irr. comp. with multisegment $\mathfrak{m} = \sum_{i=1}^{r}[a_i, b_i]$. Apply the RSK correspondence to $(a_i, b_i)_{i=1}^{r}$ to obtain a pair (P, Q) of "semistandard" Young tableaux of the same shape. The entries of P are the a_i's and the entries of Q are the b_i's. In our conventions, the entries along each row (of both P or Q) are strictly decreasing while the entries down each column are weakly decreasing.

Note that we do not get all such pairs (P, Q) because of the restriction $a_i \leq b_i$. Let k be the number of rows of P and Q and for each $i = 1, \dots, k$ let C_i be the special irr. comp. with multisegment $\sum_{j=1}^{n_i}[p_{i,j}, q_{i,j}]$ formed by the entries of the i-th row of P and Q. (Indeed, $p_{i,j} \leq q_{i,j}$.)

Theorem (Max Gurevich+•, 2020)

π_C is a subrepresentation of $\Pi_C := \pi_{C_k} \times \cdots \times \pi_{C_1}$

In fact, Gurevich proved that the socle of Π_C is irreducible (hence equal to π_C) and occurs with multiplicity one in $\mathrm{JH}(\Pi_C)$.

Upper triangularity

We can think of Π_C as a new (?) kind of standard module. Define a partial order on "semistandard" Young tableaux by

$$Y \leq Y' \quad \text{if } \operatorname{shape}(Y_{\geq r}) \prec \operatorname{shape}(Y'_{\geq r}) \text{ for all } r \geq 0 \,,$$

where $\operatorname{shape}(Y_{\geq r})$ is the Young diagram of the sub "semistandard" tableaux consisting of the entries $\geq r$ and $\prec$ is the dominance order

$$(\lambda_1, \ldots, \lambda_k) \prec (\lambda'_1, \ldots, \lambda'_{k'}) \text{ if } k \leq k' \text{ and } \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \lambda'_i \ \forall j$$

Conjecture 4

Suppose that $\pi_{C'}$ is an irreducible subquotient of Π_C. Let (P', Q') be the RSK of the corresponding multisegment. Then $P' \leq P$ and $Q' \leq Q$.

A family of standard modules

We can enhance this construction as follows. Fix a "dummy" multisegment $\mathfrak{d} = \sum_{i=1}^{l} [t_i, t_i - 1]$ with $1 \leq t_i \leq n$ and apply RSK to $\mathfrak{m} + \mathfrak{d}$. The previous theorem is still valid. We get standard modules $\Pi_C^{\mathfrak{d}}$. For an appropriate choice of $\mathfrak{d}$, $\Pi_C^{\mathfrak{d}}$ can be either the Zelevinsky standard module or the Langlands standard module. Thus, we get an interpolation between the two.
I do not know what lies behind this construction.