Unipotent Representations of Real Reductive Groups

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Plan

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- Also known: algorithm for $\widehat{G_{\mathbb{R}}}$ (Atlas, 2000s)

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- Right arrow: geometric quantization, left arrow: classical limit

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- RHS: 'looks like' functions on $\mathcal{O}(p, \nu)$
- Caveats:

$$\text{more complicated when } \mathcal{O}(p,\nu) \text{ nilpotent}$$

$$\mathcal{O}(p,\nu) \implies \text{compact quotient } \mathcal{O}(p,\nu)/\sim$$

$$\text{functions} \implies L^2 \text{ sections of Hermitian line bundle}$$

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- Problem of unitary dual 'reduces' to

Problem

Find a natural correspondence

$$\mathcal{O} = \text{nilptnt orbt} \rightsquigarrow \mathrm{Unip}(\mathcal{O}) = \text{fin set of irred unitary reps}$$

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- And a maximal ideal

$$I(\mathcal{O}^\vee) = I(\frac{1}{2}h^\vee) \subset U(\mathfrak{g})$$

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Definition (Arthur, Barbasch-Vogan)

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If $\mathcal{O} = \text{principal}$, then $\mathrm{Unip}^a(\mathcal{O}) = \{\text{irred reps of infl char } 0\}$. Includes prin series $\mathrm{Ind}_{B_{\mathbb{R}}}^{G_{\mathbb{R}}}\mathbb{C}$.

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2. Smaller problem: $\mathrm{Unip}^a(\mathcal{O})$ is too small, even when $\mathcal{O}$ is special.

Example

Let $G_{\mathbb{R}} = SL_n(\mathbb{C})$. Then $\mathrm{Unip}^a(\mathcal{O}^{\mathrm{prin}})$ consists of infl char $= 0$. Unitary rep of infl char $(\frac{n-1}{2n}, \frac{n-3}{2n}, ..., \frac{1-n}{2n})$ attached to $\mathcal{O}^{\mathrm{prin}}$, qzation of universal cover.

Unipotent Representations: Vogan

- If $I \subset U(\mathfrak{g})$,

$$U(\mathfrak{g})/I = \text{fin gen } U(\mathfrak{g}) - \text{mod}$$
$$\mathrm{gr}\,(U(\mathfrak{g})/I) = \text{fin gen } S(\mathfrak{g}) - \text{mod}$$
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Definition (Vogan)

Infl char λ is *weakly unipotent* for $\mathcal{O}$ if *rational* for wt lattice and

$$\lambda = \min\{\mu \in \lambda + \text{wt lattice} : V(I(\mu)) = \overline{\mathcal{O}}\}$$

Then

$$\mathrm{Unip}^w(\mathcal{O}) = \{X \text{ irred} \mid \mathrm{Ann}(X) = I(\lambda),\ \lambda = \text{wkly unptnt}\}$$

Unipotent Representations: Vogan

Example

Let $G_{\mathbb{R}} = SL_2(\mathbb{R})$. Then

$$\begin{aligned} \text{weight lattice} &= \mathbb{Z} \\ \text{roots} &= \{\pm 2\} \\ \text{fundamental chamber} &= \mathbb{R}_{\geq 0} \\ V(I(\lambda)) &= \begin{cases} \{0\} & \text{if } \lambda \in \{1,2,3,\ldots\} \\ \mathcal{N} & \text{else} \end{cases} \end{aligned}$$

Hence

$$\mathrm{Unip}^w(\mathcal{O}^{\mathrm{prin}}) \leftrightsquigarrow \lambda \in [0, \frac{1}{2}] \cap \mathbb{Q}$$

$$\mathrm{Unip}^w(0) \leftrightsquigarrow \lambda = 1$$

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If $G_{\mathbb{R}} = SL_2(\mathbb{R})$, $\mathcal{O} =$ principal, weak unipotent infl chars for $\mathcal{O}$ are: $[0, \frac{1}{2}] \cap \mathbb{Q}$.

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- Problem: many reps are non-unitary

Example

If $G_{\mathbb{R}} = SL_2(\mathbb{R})$, two irreps at infl char $\frac{1}{2}$. One unitary, one non-unitary.

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Theorem (L-MB-M)

In classical types,

$$\mathrm{Unip}^a(\mathcal{O}) \subset \mathrm{Unip}(\mathcal{O}) \subset \mathrm{Unip}^w(\mathcal{O})$$

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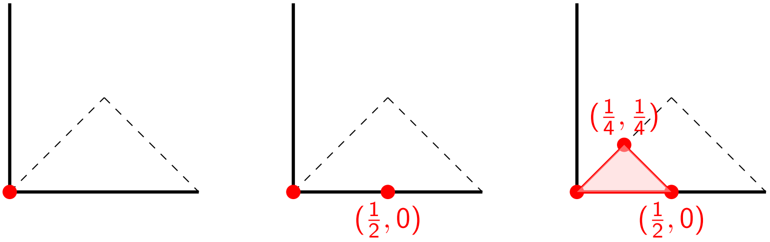
$\mathrm{Unip}^{*}(\mathcal{O}^{\mathrm{prin}})$ for $Sp(4,\mathbb{R})$:



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- In my thesis I study restriction to $K_{\mathbb{R}}$ of unipotent representations

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Lusztig-Spaltenstein:

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3. If $\mathcal{O}_{\mathfrak{g}}$ is **principal**, complete classification of $\mathrm{Unip}^a(\mathcal{O}_{\mathfrak{g}})$ and determination of K-types

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- **Kostant**:

$$S(G_{\mathbb{R}}) \simeq_K \mathbb{C}[\mathcal{N} \cap \mathfrak{p}]$$

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A unipotent parameter for $\mathcal{O}^{\mathrm{prin}}$ is a pair $(\mathfrak{q}, \chi)$, where

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3. $\mathfrak{u} \cap \mathfrak{p}$ contains a principal nilpotent element

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2. $d\chi = -\rho(\mathfrak{u})$
3. $\mathfrak{u} \cap \mathfrak{p}$ contains a principal nilpotent element

Theorem (MB)

There is a bijection

$$\mathrm{Param}(\mathcal{O}^{\mathrm{prin}})/K \simeq \mathrm{Unip}^a(\mathcal{O}^{\mathrm{prin}}) \quad (\mathfrak{q}, \chi) \mapsto \mathrm{CohInd}_{\mathfrak{q}}^{\mathfrak{g}}(\chi \otimes S)$$

Principal Unipotent Reps: Proof Sketch

- **Harish-Chandra**: category equivalence

$$\mathrm{HC} : \mathrm{Rep}^{\lambda}(G_{\mathbb{R}}) \simeq M^{\lambda}(\mathfrak{g}, K)$$

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- **Beilinson-Bernstein**: quotient functor

$$\Gamma : M(\mathcal{D}^{\lambda}, K) \twoheadrightarrow M^{\lambda}(\mathfrak{g}, K)$$

Equivalence if λ regular.

Principal Unipotent Reps: Proof Sketch

- Hence:

$$\mathrm{Unip}^{a}(\mathcal{O}^{\mathrm{prin}}) \simeq \{\text{irred } \mathcal{M} \in M(\mathcal{D}^0, K) : \Gamma(\mathcal{M}) \neq 0\}$$

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$$\{\text{irreds in } M(\mathcal{D}^0, K)\} \simeq \{(Z \overset{K-\text{orbt}}{\subset} \mathcal{B}, \gamma \overset{\text{loc sys}}{\to} Z)\}$$

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- Choose a base point:

$$\{(Z,\gamma)\} \simeq \{(\mathfrak{b} \subset \mathfrak{g}, \chi \in \widehat{H_{\mathbb{R}}}) : d\chi = -\rho(\mathfrak{n})\}/K$$

and

$$\Gamma(\mathcal{B}, j_!\gamma) \simeq \mathrm{Ind}_{\mathfrak{b}, H\cap K}^{\mathfrak{g}, K} \chi$$

Principal Unipotent Reps: Proof Sketch

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Principal Unipotent Reps: Proof Sketch

- The non-vanishing condition $\Gamma(\mathcal{B}, j_!\gamma) \neq 0$ imposes a constraint in each simple root direction
- These conditions precisely guarantee: $(\mathfrak{b}, \chi)$ comes from a unipotent parameter $(\mathfrak{q}, \chi)$
- In this case, $j_!\gamma$ is *irreducible* and

$$\begin{aligned}\Gamma(\mathcal{B}, j_!\gamma) &= \mathrm{Ind}_{\mathfrak{b}, H\cap K}^{\mathfrak{g}, K}\chi \\ &= \mathrm{Ind}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, K}\mathrm{Ind}_{\mathfrak{l}\cap\mathfrak{b}, H\cap K}^{\mathfrak{l}, L\cap K}\chi \\ &= \mathrm{CohInd}_{\mathfrak{q}}^{\mathfrak{g}}\chi \otimes S(L_{\mathbb{R}})\end{aligned}$$

Principal Unipotent Reps: K-types

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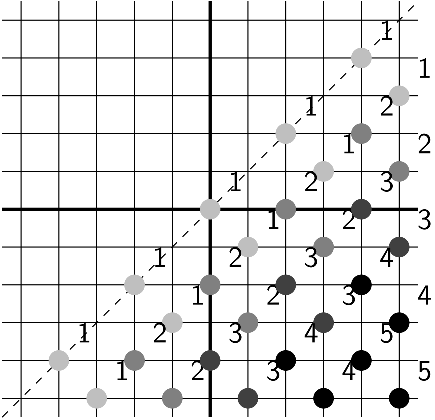
- Nice feature of this classification: K-structure is 'visible'
- **Kostant**: $S \simeq_K \mathbb{C}[\mathcal{N} \cap \mathfrak{p}]$
- **Blattner**: gives K-structure of cohomological induction
- Combined: we get nice formulas for K-types of $\mathrm{Unip}^a(\mathcal{O}^{\mathrm{prin}})$ which are computable using Borel-Weil-Bott

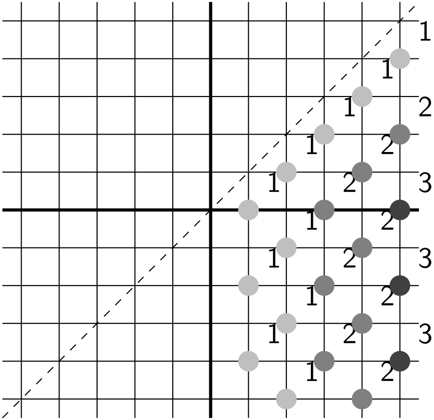
Principal Unipotent Reps: $Sp(4, \mathbb{R})$

Five elements of $\mathrm{Unip}^a(\mathcal{O}^{\mathrm{prin}})$: principal series S, two limit of discrete series, and two 'in-between'



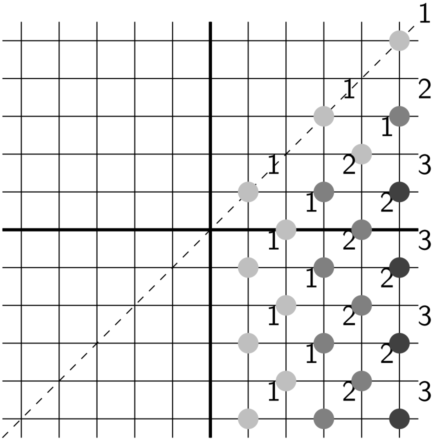
Spherical principal series (quantization of $\mathcal{N} \cap \mathfrak{p}$)

Principal Unipotent Reps: $Sp(4, \mathbb{R})$



Limit of discrete series (quantization of one irred component of $\mathcal{N} \cap \mathfrak{p}$)

Principal Unipotent Reps: $Sp(4, \mathbb{R})$



Cohomologically induced from spherical principal series of Segal parabolic (quantization of one irred component of $\mathcal{N} \cap \mathfrak{p}$)