Arthur packets for G_2 and perverse sheaves on cubics

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joint work with Andrew Fiori and Qing Zhang

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https://mathseminars.org/talk/MITLie/3/

Abstract

This talk demonstrates a non-invasive procedure that calculates Arthur packets, their associated stable distributions and Langlands-Shelstad transfers, without direct use of endoscopy, using certain unipotent representations of the split p-adic exceptional group G_2 as examples. In the case at hand, this procedure relies on a study of the category of GL_2-equivariant perverse sheaves on the moduli space of homogeneous cubics in two variables, which is perhaps of independent interest. Specifically, we find the Fourier transform and the microlocalization of the simple objects in this category, and convert that into information about the Aubert involution and stable distributions attached to Arthur packets.

This is joint work with Andrew Fiori and Qing Zhang, based on earlier joint work with Andrew Fiori, Ahmed Moussaoui, James Mracek and Bin Xu, which is based on earlier work by David Vogan, sadly, not joint.

Motivation

From [Art13]:

THEOREM 1.5.1. *Assume that F is local and that $G \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$.*

(a) *For any local parameter $\psi \in \widetilde{\Psi}(G)$, there is a finite set $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$, constructed from ψ by endoscopic transfer, and equipped with a canonical mapping*

$$\pi \longrightarrow \langle \cdot, \pi \rangle, \qquad \pi \in \widetilde{\Pi}_\psi,$$

from $\widetilde{\Pi}_\psi$ into the group $\widehat{\mathcal{S}}_\psi$ of characters on $\mathcal{S}_\psi$ such that $\langle \cdot, \pi \rangle = 1$ if G and π are unramified (relative to K_F).

(b) *If $\phi = \psi$ belongs to the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of parameters in $\widetilde{\Psi}(G)$ that are trivial on the factor $SU(2)$, the elements in $\widetilde{\Pi}_\phi$ are tempered and multiplicity free, and the corresponding mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is injective. Moreover, every element in $\widetilde{\Pi}_{\mathrm{temp}}(G)$ belongs to exactly one packet $\widetilde{\Pi}_\phi$. Finally, if F is nonarchimedean, the mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is bijective.*

We wish to adapt this theorem to the p-adic group $G_2(F)$.

Strategy

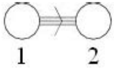
1. Define candidate packets using a microlocal analysis of a moduli space of Langlands parameters.
2. Show that they enjoy the defining properties that we see in the case of orthogonal and symplectic groups, in this case, for the elliptic endoscopic groups PGL_3 and SO_4 and for G_2 as a twisted endoscopic group of PSO_8;
3. Attach stable distributions to these candidate packets.

Only then should the candidate packets be called Arthur packets!

In this talk, and the paper by the same title that will appear on the arXiv tomorrow, we take the first step in the most interesting case for $G_2(F)$ and give a partial result on the third step.
The second step is treated in forthcoming work.

The group G_2

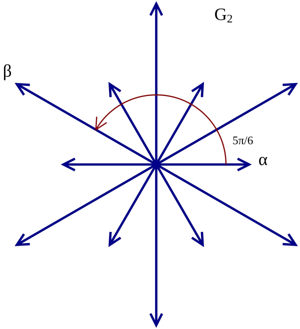
We view G_2 as a Chevalley group scheme with Dynkin diagram



with long root β and short root α. There are 6 positive roots

$$\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

The root system is given by



Subregular Langlands parameter

- There are five unipotent conjugacy classes in $\widehat{G}_2$:

$$\mathcal{O}_0 < \mathcal{O}_{\rm long} < \mathcal{O}_{\rm short} < \mathcal{O}_{\rm sub} < \mathcal{O}_{\rm reg}.$$

- Each unipotent conjugacy classes determines a group homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_2$. In particular, we have

$$\varphi_{\rm sub} : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_2.$$

- Consider the Langlands parameter $\phi_3 : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_2$ defined by

$$\phi_3(w, x) := \varphi_{\rm sub}(x).$$

(Notation will be justified later.)

- The component group of ϕ_3 is non-Abelian:

$$A_{\phi_3} := \pi_0(Z_{\widehat{G}_2}(\phi_3)) \cong S_3.$$

The "superunipotent" representation of G_2

- The L-packet for ϕ_3 is

$$\Pi_{\phi_3}(G_2(F)) = \{\pi_3, \pi_3^{\varrho}, \pi_3^{\varepsilon}\},$$

where
  - π_3 corresponds to the trivial representation of S_3,
  - π_3^{ϱ} corresponds to the unique 2-dimensional irreducible representation ϱ of S_3, and
  - π_3^{ε} corresponds to the sign character ε of S_3.
- π_3^{ε} is supercuspidal:

$$\pi_3^{\varepsilon} := \mathrm{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)} G_2[1];$$

where $G_2[1]$ is a cuspidal unipotent representation of $G_2(\mathbb{F}_q)$.
- All three unipotent representations are tempered. Both π_3 and π_3^{ϱ} have nonzero Iwahori-fixed vectors.

The category of subregular unipotent representations

- This L-packet straddles two blocks in the Bernstein decomposition of the category of smooth representations of $G_2(F)$:

$$\mathrm{Rep}(G_2(F)) = \mathrm{Rep}(G_2(F))_{[T(F),\mathbb{1}]} \oplus \cdots \oplus \mathrm{Rep}(G_2(F))_{[G_2(F),\pi_3^{\varepsilon}]}$$

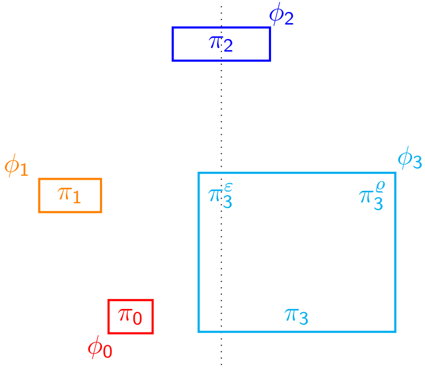
- We can refine this by specifying the relevant cuspidal support, not just the inertial classes. There is a unique unramified character $\chi_{\mathrm{sub}} : T(F) \to \mathbb{C}^\times$ such that

$$\mathrm{Rep}(G_2(F))_{\mathrm{sub}} := \mathrm{Rep}(G_2(F))_{(T(F),\chi_{\mathrm{sub}})} \oplus \mathrm{Rep}(G_2(F))_{(G_2(F),\pi_3^{\varepsilon})}$$

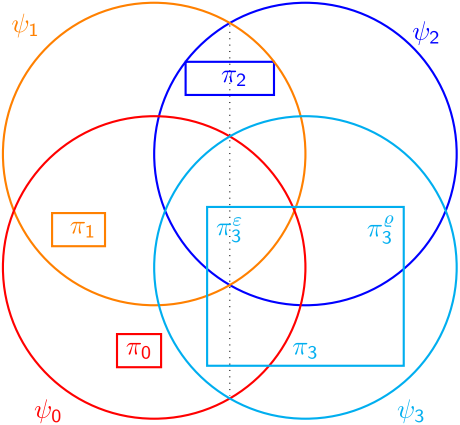
  contains π_3 and π_3^{ϱ} in the first block and π_3^{ε} in the second.
- We can identify all six irreducible representations in this category:

$$\left(\mathrm{Rep}(G_2(F))_{\mathrm{sub}}\right)^{\mathrm{irred}}_{/\mathrm{equiv}} = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_3^{\varrho}, \pi_3^{\varepsilon}\}.$$

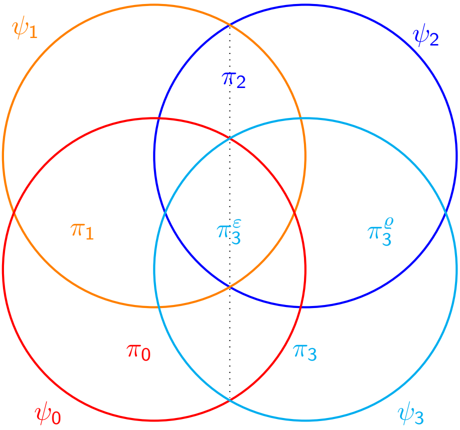
$\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$ partitioned into L-packets



$\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$ decomposed into A-packets

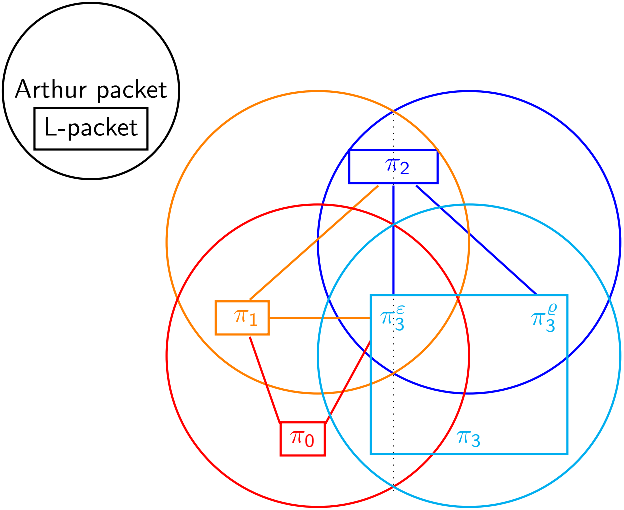


A-packets and Aubert duality on $K\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$

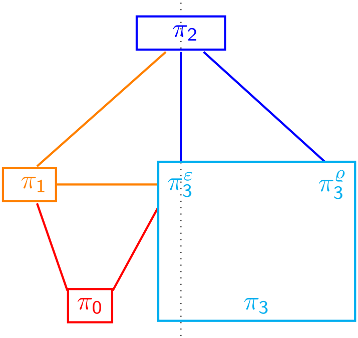


Aubert duality $D_{G_2} : K\mathrm{Rep}(G_2(F))_{\mathrm{sub}} \to K\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$ acts by reflection around the dotted line.

Stabilizing representations



Stabilizing representations and stable distributions



$$
\begin{aligned}
\Theta_{\psi_0} &:= \Theta_{\pi_0} + 2\Theta_{\pi_1} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_1} &:= \Theta_{\pi_1} - \Theta_{\pi_2} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_2} &:= \Theta_{\pi_2} - \Theta_{\pi_3^\varrho} - \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_3} &:= \Theta_{\pi_3} + 2\Theta_{\pi_3^\varrho} + \Theta_{\pi_3^\varepsilon}
\end{aligned}
$$

Main result

Theorem

For each Arthur parameter $\psi : W'_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_2$ *of* $G_2(F)$ *with subregular infinitesimal parameter there exists a finite set* $\Pi_\psi(G_2(F))$ *of irreducible unipotent representations and a function*

$$\begin{array}{rcl} \langle\ ,\ \rangle_\psi : \Pi_\psi(G_2(F)) & \to & \mathrm{IrrRep}(A_\psi) \\ \pi & \mapsto & \langle\ ,\pi\rangle_\psi, \end{array}$$

where $A_\psi := \pi_0(Z_{\widehat{G}_2}(\psi))$ *, such that*

(a) *if* ψ *is trivial on* $\mathrm{SL}_2(\mathbb{C})$ *then* $\langle\ ,\rangle_\psi$ *is bijective and all the representations in* $\Pi_\psi(G_2(F))$ *are tempered;*

(b) *if* ψ *is not trivial on* $\mathrm{SL}_2(\mathbb{C})$ *then* $\langle\ ,\ \rangle_\psi$ *is not bijective and* $\Pi_\psi(G_2(F))$ *contains non-tempered representations;*

(c) *if* π *is spherical and* $\pi \in \Pi_\psi(G_2(F))$ *then* $\langle\ ,\pi\rangle_\psi = \mathbb{1}$*, the trivial representation of* A_ψ*.*

L-packets and A-packets for all irreducible representations in $\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$

A − packet	L − packet
$\Pi_{\psi_0}(G_2(F)) = \{\pi_0, \pi_1, \pi_3^{\varepsilon}\}$	$\Pi_{\phi_0}(G_2(F)) = \{\pi_0\}$
$\Pi_{\psi_1}(G_2(F)) = \{\pi_1, \pi_2, \pi_3^{\varepsilon}\}$	$\Pi_{\phi_1}(G_2(F)) = \{\pi_1\}$
$\Pi_{\psi_2}(G_2(F)) = \{\pi_2, \pi_3^{\varrho}, \pi_3^{\varepsilon}\}$	$\Pi_{\phi_2}(G_2(F)) = \{\pi_2\}$
$\Pi_{\psi_3}(G_2(F)) = \{\pi_3, \pi_3^{\varrho}, \pi_3^{\varepsilon}\}$	$\Pi_{\phi_3}(G_2(F)) = \{\pi_3, \pi_3^{\varrho}, \pi_3^{\varepsilon}\}$

$$\begin{aligned} A_{\psi_0} &= S_3, & A_{\phi_0} &= 1, \\ A_{\psi_1} &= S_2, & A_{\phi_1} &= 1 \\ A_{\psi_2} &= S_2, & A_{\phi_2} &= 1 \\ A_{\psi_3} &= S_3, & A_{\phi_3} &= S_3. \end{aligned}$$

The functions $\langle\ ,\ \rangle_\psi : \Pi_\psi(G_2(F)) \to \mathrm{Rep}(A_\psi)$ appearing in the main result

$\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$	$\mathrm{Rep}(A_{\psi_0})$	$\mathrm{Rep}(A_{\psi_1})$	$\mathrm{Rep}(A_{\psi_2})$	$\mathrm{Rep}(A_{\psi_3})$
π_0	$\mathbb{1}$	0	0	0
π_1	ϱ	$\mathbb{1}$	0	0
π_2	0	τ	$\mathbb{1}$	0
π_3	0	0	0	$\mathbb{1}$
π_3^ϱ	0	0	τ	ϱ
π_3^ε	ε	$\mathbb{1}$	τ	ε

$$\begin{aligned}
A_{\psi_0} &= S_3, & \mathrm{Irrep}(A_{\psi_0}) &= \{\mathbb{1}, \varrho, \varepsilon\}\\
A_{\psi_1} &= S_2, & \mathrm{Irrep}(A_{\psi_1}) &= \{\mathbb{1}, \tau\}\\
A_{\psi_2} &= S_2, & \mathrm{Irrep}(A_{\psi_2}) &= \{\mathbb{1}, \tau\}\\
A_{\psi_3} &= S_3, & \mathrm{Irrep}(A_{\psi_3}) &= \{\mathbb{1}, \varrho, \varepsilon\}
\end{aligned}$$

Stable distributions

Theorem

For every Arthur parameter ψ with subregular infinitesimal parameter, consider the invariant distribution

$$\Theta_\psi := \sum_{\pi\in\Pi_\psi(G_2(F))} \langle a_\psi, \pi\rangle_\psi \ \Theta_\pi ,$$

where a_ψ is the image of $s_\psi := \psi(1,-1)$ in A_ψ. Suppose Θ_ψ is stable when ψ is tempered.[1] *Then the distributions Θ_ψ are stable for all Arthur parameters ψ.*

$$\begin{aligned}
\Theta_{\psi_0} &= \Theta_{\pi_0} + 2\Theta_{\pi_1} + \Theta_{\pi_3^\varepsilon}\\
\Theta_{\psi_1} &= \Theta_{\pi_1} - \Theta_{\pi_2} + \Theta_{\pi_3^\varepsilon}\\
\Theta_{\psi_2} &= \Theta_{\pi_2} - \Theta_{\pi_3^\varrho} - \Theta_{\pi_3^\varepsilon}\\
\Theta_{\psi_3} &= \Theta_{\pi_3} + 2\Theta_{\pi_3^\varrho} + \Theta_{\pi_3^\varepsilon}
\end{aligned}$$

[1] We hope to remove this hypothesis in future work.

Subregular infinitesimal parameter

- The "infinitesimal parameter" $\lambda_\phi : W_F \to \widehat{G}_2$ of a Langlands parameter ϕ is defined by

$$\lambda_\phi(w) := \phi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right).$$

  Set $\lambda_{\text{sub}} := \lambda_{\phi_3}$; we refer to this as the subregular infinitesimal parameter.

- Up to $\widehat{G}_2$-conjugacy, there are four Langlands parameters with infinitesimal parameter λ_{sub}, each of Arthur type:

$$\begin{aligned} \psi_0(w,x,y) &= \varphi_{\text{sub}}(y), \\ \psi_1(w,x,y) &= \iota_{\hat\alpha}(x)\ \iota_{\hat\alpha+2\hat\beta}(y), \\ \psi_2(w,x,y) &= \iota_{\hat\alpha}(y)\ \iota_{\hat\alpha+2\hat\beta}(x), \\ \psi_3(w,x,y) &= \varphi_{\text{sub}}(x), \end{aligned}$$

  where $\iota_{\hat\alpha} : \mathrm{SL}_2 \to \widehat{G}_2$ embeds into Levi subgroup $\mathrm{GL}_2(\hat\alpha)$ and $\iota_{\hat\alpha+2\hat\beta} : \mathrm{SL}_2 \to \widehat{G}_2$ embeds into Levi subgroup $\mathrm{GL}_2(\hat\alpha + 2\hat\beta)$.

Langlands Correspondence

The local Langlands correspondence for representations with subregular infinitesimal parameter:

enhanced L − parameter		irreducible admissible rep′n
$(\phi_0, 1)$	$\mapsto$	π_0
$(\phi_1, 1)$	$\mapsto$	π_1
$(\phi_2, 1)$	$\mapsto$	π_2
$(\phi_3, 1)$	$\mapsto$	π_3
(ϕ_3, ϱ)	$\mapsto$	π_3^{ϱ}
(ϕ_3, ε)	$\mapsto$	π_3^{ε}

Moduli space of Langlands parameters

The moduli space of Langlands parameters with subregular infinitesimal parameter is

$$X_{\mathrm{sub}} = \left(\widehat{G}_2 \times V_{\mathrm{sub}}\right) / H_{\mathrm{sub}}$$

where

$$V_{sub} := \{x \in \widehat{\mathfrak{g}}_2 \mid \mathrm{Ad}(\lambda_{sub}(\mathrm{Fr}))x = qx\} = \left\{\sum_{i=0}^{3} r_i X_{\hat{\alpha}+i\hat{\beta}} \mid r_i \in \mathbb{C}\right\}$$

and

$$H_{sub} := Z_{\widehat{G}_2}(\lambda_{\mathrm{sub}}) \cong \mathrm{GL}_2(\hat{\beta}),$$

Langlands parameters as homogeneous cubics in two variables

Let $P_3[x, y]$ be the vector space of homogeneous cubics in two variables x, y.

Proposition

There is a natural isomorphism $V_{\mathrm{sub}} \to P_3[x, y]$ that defines an equivalence of representations of H_{sub} on V_{sub} with the representation $\det^{-1} \otimes \mathrm{Sym}^3$ of GL_2 on $P_3[x, y]$ given below.

Consequently,

$$\mathrm{Per}_{\widehat{G}_2}(X_{\mathrm{sub}}) \cong \mathrm{Per}_{H_{sub}}(V_{\mathrm{sub}}) \cong \mathrm{Per}_{\mathrm{GL}_2}(\det{}^{-1} \otimes \mathrm{Sym}^3)$$

Sym^3

The action

$$\det{}^{-1} \otimes \mathrm{Sym}^3 : \mathrm{GL}_2 \to \mathrm{Aut}(P_3[x,y])$$

is defined as follows: for $h \in \mathrm{GL}_2$ and $r \in P_3[x,y]$,

$$\begin{aligned} &((\det{}^{-1} \otimes \mathrm{Sym}^3)(h).r)(x,y) \\ := \quad & \det(h)^{-1} r((x,y)h) \\ = \quad & \det(h)^{-1} r(ax+cy, bx+dy), \qquad h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \end{aligned}$$

A little geometry

- The GL_2 action $\det^{-1} \otimes \mathrm{Sym}^3$ on $P_3[x,y]$ has the following 4 orbits:
  - $C_0 = \{0\}$;
  - $C_1 = \left\{u^3 \mid u \in P_1[x,y], p \neq 0\right\}$;
  - $C_2 = \left\{u^2 u' \mid u, u' \in P_1[x,y], \text{ and } u, u' \text{ are linearly indep.}\right\}$;
  - $C_3 = \left\{u u' u'' \mid u, u', u'' \in P_1[x,y], \text{ and } u, u', u'' \text{ are indep.}\right\}$.
- The closure relation on the GL_2-orbits in $P_3(x,y)$ is given by

$$C_0 < C_1 < C_2 < C_3.$$

- Component groups/fundamental groups and microlocal fundamental groups:

$$\begin{aligned} A_{\phi_0} &= A_{C_0} = 1 & A_{\psi_0} &= A^{\mathrm{mic}}_{C_0} = S_3 \\ A_{\phi_1} &= A_{C_1} = 1 & A_{\psi_1} &= A^{\mathrm{mic}}_{C_1} = S_2 \\ A_{\phi_2} &= A_{C_2} = 1 & A_{\psi_2} &= A^{\mathrm{mic}}_{C_2} = S_2 \\ A_{\phi_3} &= A_{C_3} = S_3 & A_{\psi_3} &= A^{\mathrm{mic}}_{C_3} = S_3 \end{aligned}$$

A little geometry

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  - $C_3 = \{uu'u'' \mid u, u', u'' \in P_1[x,y], \text{ and } u, u', u'' \text{ are indep.}\}$.
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Langlands correspondence, revisited

$\mathcal{P} : \left(\mathrm{Rep}(G_2(F))\right)^{\text{irred}}_{/\text{equiv}}$	$\to$	$\left(\mathrm{Per}_{\mathrm{GL}_2}(\det{}^{-1} \otimes \mathrm{Sym}^3)\right)^{\text{simple}}_{/\text{iso}}$
π_0	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{C_0})$
π_1	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{C_1})$
π_2	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{C_2})$
π_3	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{C_3})$
π_3^{ϱ}	$\mapsto$	$\mathcal{IC}(\mathcal{R}_{C_3})$
π_3^{ε}	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{C_3})$

where:

- $\mathbb{1}_{C_0}$ is the trivial local system on C_0, $\mathbb{1}_{C_1}$ is the trivial local system on C_1, $\mathbb{1}_{C_2}$ is the trivial local system on C_2, $\mathbb{1}_{C_3}$ is the trivial local system on C_3;
- $\mathcal{R}_{C_3}$ is the GL_2-equivariant local system on C_3 corresponding to representation ϱ of $A_{C_3} \cong A_{\phi_3} \cong S_3$;
- $\mathcal{E}_{C_3}$ is the GL_2-equivariant local system on C_3 corresponding to the representation ε of $A_{C_3} \cong A_{\phi_3} \cong S_3$.

Microlocal vanishing cycles

For any algebraic group H acting on $V \cong \mathbb{A}^d$ with finitely-many orbits, the microlocal vanishing cycles functor

$$\mathsf{Evs} : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(\Lambda^{\mathrm{reg}})$$

is defined [CFM+21] by

$$\mathsf{Evs}_{C_i} \mathcal{F} := \left(\mathsf{R}\Phi_{(\ |\ )}[-1]\mathcal{F} \boxtimes \mathbb{1}^!_{C_i^*}[\dim C_i^*] \right) |_{\Lambda_i^{\mathrm{reg}}}[-4],$$

where $\Lambda := T^*_H(V)$ is the conormal variety and $\Lambda_i := T^*_{C_i} V$. Then

$$\Lambda^{\mathrm{reg}} = \bigcup_i \Lambda_i^{\mathrm{reg}} \qquad \text{(disjoint union)},$$

so Evs may be decomposed according to

$$\mathsf{Evs} = \oplus_i \mathsf{Evs}_{C_i} .$$

The main geometric result

Theorem

On simple objects, Evs *is given by:*

$$
\begin{array}{ccl}
\mathrm{Per}_{\mathrm{GL}_2}(\det^{-1}\otimes \mathrm{Sym}^3) & \xrightarrow{\mathsf{Evs}} & \mathrm{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathbb{1}_{\Lambda_0^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto & \mathcal{T}_{\Lambda_1^{\mathrm{reg}}} \oplus \mathcal{R}_{\Lambda_0^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{T}_{\Lambda_2^{\mathrm{reg}}} \oplus \mathbb{1}_{\Lambda_1^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathbb{1}_{\Lambda_3^{\mathrm{reg}}} \\
\mathcal{IC}(\mathcal{R}_{C_3}) & \mapsto & \mathcal{R}_{\Lambda_3^{\mathrm{reg}}} \oplus \mathbb{1}_{\Lambda_2^{\mathrm{reg}}} \\
\mathcal{IC}(\mathcal{E}_{C_3}) & \mapsto & \mathcal{R}_{\Lambda_3^{\mathrm{reg}}} \oplus \mathcal{T}_{\Lambda_2^{\mathrm{reg}}} \oplus \mathbb{1}_{\Lambda_1^{\mathrm{reg}}} \oplus \mathcal{E}_{\Lambda_0^{\mathrm{reg}}},
\end{array}
$$

where

- $\mathbb{1}_{\Lambda_i^{\mathrm{reg}}}$ *is the constant local system on* Λ_i^{reg} *for* $i = 0, 1, 2, 3$*;*
- $\mathcal{T}_{\Lambda_1^{\mathrm{reg}}}$ *(resp.* $\mathcal{T}_{\Lambda_2^{\mathrm{reg}}}$*) is the local system on* Λ_1^{reg} *(resp.* Λ_2^{reg}*) for the non-trivial character* τ *of* $A^{\mathrm{mic}}_{C_1} = S_2$ *(resp.* $A^{\mathrm{mic}}_{C_2} = S_2$*);*
- $\mathcal{R}_{\Lambda_3^{\mathrm{reg}}}$ *corresponds to the representation* ϱ *of* $A^{\mathrm{mic}}_{C_3} = S_3$*;*
- $\mathcal{E}_{\Lambda_3^{\mathrm{reg}}}$ *corresponds to the representation* ε *of* $A^{\mathrm{mic}}_{C_3} = S_3$*.*

Evs calculated

$$\mathsf{Evs} : \mathsf{Per}_{\mathrm{GL}_2}(\det{}^{-1} \otimes \mathrm{Sym}^3) \longrightarrow \mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{reg}})$$

$\mathsf{Per}_{\mathrm{GL}_2}(P_3[x,y])$	$\mathsf{Loc}(\Lambda_0^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_1^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_2^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_3^{\mathrm{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda_0^{\mathrm{reg}}}$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{R}_{\Lambda_0^{\mathrm{reg}}}$	$\mathcal{T}_{\Lambda_1^{\mathrm{reg}}}$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	0	$\mathbb{1}_{\Lambda_1^{\mathrm{reg}}}$	$\mathcal{T}_{\Lambda_2^{\mathrm{reg}}}$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$\mathbb{1}_{\Lambda_3^{\mathrm{reg}}}$
$\mathcal{IC}(\mathcal{R}_{C_3})$	0	0	$\mathbb{1}_{\Lambda_2^{\mathrm{reg}}}$	$\mathcal{R}_{\Lambda_3^{\mathrm{reg}}}$
$\mathcal{IC}(\mathcal{E}_{C_3})$	$\mathcal{E}_{\Lambda_0^{\mathrm{reg}}}$	$\mathcal{T}_{\Lambda_1^{\mathrm{reg}}}$	$\mathbb{1}_{\Lambda_2^{\mathrm{reg}}}$	$\mathcal{E}_{\Lambda_3^{\mathrm{reg}}}$

Definition

$$\Pi_{\psi_i}(G_2(F)) := \left\{ \pi \in \Big(\mathsf{Rep}(G_2(F))\Big)^{\mathrm{irred}}_{/\mathrm{equiv}} \;\middle|\; \mathsf{Evs}_{C_i}\,\mathcal{P}(\pi) \neq 0 \right\}$$

Normalization/Twisting (by the orientation sheaf?)

From [CFM+21, Section 7.10], we recall the normalized microlocal vanishing cycles functor

$$\mathsf{NEvs}_C : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(T^*_C\Lambda^{\mathrm{reg}}),$$

defined by

$$\mathsf{NEvs}_C\, \mathcal{F} := \mathcal{H}\mathrm{om}\left(\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_C), \mathsf{Evs}_C\, \mathcal{F}\right) = \left(\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_C)\right)^\vee \otimes \mathsf{Evs}_C\, \mathcal{F}.$$

In our case,

$$\mathsf{Evs}_{C_i}\, \mathcal{IC}(\mathbb{1}_{C_i}) = \begin{cases} \mathbb{1}_{\Lambda_i^{\mathrm{reg}}}, & i = 0, 3; \\ \mathcal{T}_{\Lambda_i^{\mathrm{reg}}}, & i = 1, 2. \end{cases}$$

Normalized microlocal vanishing cycles functor

Theorem

On simple objects, the normalized microlocal vanishing cycles functor NEvs *is given by:*

$$
\begin{array}{rcl}
\mathrm{Per}_{\mathrm{GL}_2}(P_3[x,y]) & \xrightarrow{\mathrm{NEvs}} & \mathrm{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathbb{1}_{\Lambda_0^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto & \mathbb{1}_{\Lambda_1^{\mathrm{reg}}} \oplus \mathcal{R}_{\Lambda_0^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathbb{1}_{\Lambda_2^{\mathrm{reg}}} \oplus \mathcal{T}_{\Lambda_1^{\mathrm{reg}}} \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathbb{1}_{\Lambda_3^{\mathrm{reg}}} \\
\mathcal{IC}(\mathcal{R}_{C_3}) & \mapsto & \mathcal{R}_{\Lambda_3^{\mathrm{reg}}} \oplus \mathcal{T}_{\Lambda_2^{\mathrm{reg}}} \\
\mathcal{IC}(\mathcal{E}_{C_3}) & \mapsto & \mathcal{E}_{\Lambda_3^{\mathrm{reg}}} \oplus \mathbb{1}_{\Lambda_2^{\mathrm{reg}}} \oplus \mathcal{T}_{\Lambda_1^{\mathrm{reg}}} \oplus \mathcal{E}_{\Lambda_0^{\mathrm{reg}}}
\end{array}
$$

NEvs calculated

$$\mathsf{NEvs} : \mathsf{Per}_{\mathrm{GL}_2}(\det{}^{-1} \otimes \mathrm{Sym}^3) \longrightarrow \mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{reg}})$$

$\mathsf{Per}_{\mathrm{GL}_2}(P_3[x,y])$	$\mathsf{Loc}(\Lambda_0^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_1^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_2^{\mathrm{reg}})$	$\mathsf{Loc}(\Lambda_3^{\mathrm{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda_0^{\mathrm{reg}}}$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{R}_{\Lambda_0^{\mathrm{reg}}}$	$\mathbb{1}_{\Lambda_1^{\mathrm{reg}}}$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	0	$\mathcal{T}_{\Lambda_1^{\mathrm{reg}}}$	$\mathbb{1}_{\Lambda_2^{\mathrm{reg}}}$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$\mathbb{1}_{\Lambda_3^{\mathrm{reg}}}$
$\mathcal{IC}(\mathcal{R}_{C_3})$	0	0	$\mathcal{T}_{\Lambda_2^{\mathrm{reg}}}$	$\mathcal{R}_{\Lambda_3^{\mathrm{reg}}}$
$\mathcal{IC}(\mathcal{E}_{C_3})$	$\mathcal{E}_{\Lambda_0^{\mathrm{reg}}}$	$\mathbb{1}_{\Lambda_1^{\mathrm{reg}}}$	$\mathcal{T}_{\Lambda_2^{\mathrm{reg}}}$	$\mathcal{E}_{\Lambda_3^{\mathrm{reg}}}$

$\langle\ ,\pi\rangle_\psi \in \mathrm{Rep}(A_\psi)$

Definition

For all Arthur parameters ψ with subregular infinitesimal parameter, for all $\pi \in \left(\mathrm{Rep}(G_2(F))\right)^{\mathrm{irred}}_{/\mathrm{equiv}}$, for all $a \in A_\psi^{\mathrm{mic}}$,

$$\langle a, \pi\rangle_\psi := (-1)^{\dim \operatorname{supp} \mathcal{P}(\pi)} (-1)^{\dim C_\psi} \operatorname{trace}_{aa_\psi} \operatorname{NEvs}_{C_\psi} \mathcal{P}(\pi),$$

where a_ψ is the image of $s_\psi := \psi(1,-1) \in Z_{\widehat{G}_2}(\psi)$ in A_ψ.

The proof of the main result now follows by direct inspection of the tables.

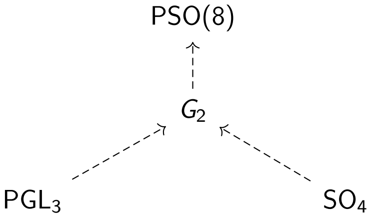
Stability

Let M_i denote the standard module for π_i for $i = 0, 1, 2$. Since M_0, M_1 and M_2 are obtained by parabolic induction from representations of $\mathrm{GL}_2(F)$, they are stable standard modules. Write Θ_{M_0}, Θ_{M_1} and Θ_{M_2} for the Harish-Chandra distribution characters attached to these representations. These distributions are stable. The distributions Θ_{ψ_i}, for $i = 0, 1, 2$ are expressed in terms of these four stable distributions, Θ_{M_0}, Θ_{M_1}, Θ_{M_2} and Θ_{ψ_3}, as follows:

$$\begin{pmatrix} \Theta_{\psi_0} \\ \Theta_{\psi_1} \\ \Theta_{\psi_2} \\ \Theta_{\psi_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{M_0} \\ \Theta_{M_1} \\ \Theta_{M_2} \\ \Theta_{\psi_3} \end{pmatrix}$$

Forthcoming work

- In a sequel to the paper for this talk, we consider the remaining unipotent representations of p-adic $G_2(F)$ and calculate the ABV-packets for these representations.
- We show that ABV-packets for all unipotent representations of G_2 are indeed Arthur packets by showing that they satisfy the conditions imposed on them by the theory of endoscopy and twisted endoscopy:



- Prove that the tempered distributions, like Θ_{ψ_3}, are stable.

A triple of "superunipotent" representations

split group	unipotent orbit	fund'l group	irred rep'n	superunipotent representation	Lusztig's class'n
G_2	$G_2(a_1)$	S_3	ε (sign)	$\mathrm{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)} G_2[1]$	7.33
F_4	$F_4(a_3)$	S_4	ε (sign)	$\mathrm{cInd}_{F_4(\mathcal{O}_F)}^{F_4(F)} F_4^{\mathrm{II}}[1]$	7.26
E_8	$E_8(a_7)$	S_5	ε (sign)	$\mathrm{cInd}_{E_8(\mathcal{O}_F)}^{E_8(F)} E_8^{\mathrm{II}}[1]$	7.1

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