COMMENTS TO THE DACHA WORKSHOP TALK

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The variety of regular elements $\mathfrak{g}^{reg}$ carries a flat group scheme Z, the family of centralizers. Using the Kostant section we get a flat group scheme over $\mathfrak{t}/W$. The vector bundle of Lie algebras is canonically identified with the cotangent bundle to $\mathfrak{t}/W$.

Thus we get a functor $Coh^Z(\mathfrak{t}/W) \to Coh(T(\mathfrak{t}/W))$, coherent sheaves on the tangent space to $\mathfrak{t}/W$. Using base change we get a functor $Coh^Z(S) \to Coh(T(\mathfrak{t}/W) \times_{\mathfrak{t}/W} S)$ where $S = \mathfrak{t} \times_{\mathfrak{t}/W} t$ is Soergel's variety (the action of Z everywhere is, of course, trivial).

There is an increasing sequence of closed subschemes Z_n in $T(\mathfrak{t}/W) \times_{\mathfrak{t}/W} S$ of dimension $\dim(\mathfrak{t})$ and the above functor lands in $Coh(Z_n)$ for some n. This is related to the category of affine Soergel bimodules as follows.

Consider the affine Cartan $\widehat{\mathfrak{t}}$ fitting in the short exact sequence

$$0 \to \mathfrak{t} \to \widehat{\mathfrak{t}} \to \mathbb{C}.$$

Let us exhaust W_{aff} by finite subsets $W_1 \subset W_2 \ldots$ and let Z'_n be the union of the graphs of $w \in W_n$, thus $Z'_n \subset \widehat{\mathfrak{t}} \times_{\mathbb{C}} \widehat{\mathfrak{t}}$. The action of W_{aff} on $\mathfrak{t}$ factors through W, so $Z'_n \cap (\mathfrak{t} \times \mathfrak{t})$ is contained in $S = \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$.

Let us now blow up S inside $\widehat{\mathfrak{t}} \times_{\mathbb{C}} \widehat{\mathfrak{t}}$ removing the proper preimage of the zero fiber $\mathfrak{t} \times \mathfrak{t}$. We get a "renormalized" version of $\widehat{\mathfrak{t}} \times_{\mathbb{C}} \widehat{\mathfrak{t}}$ which maps to $\mathbb{C}$, with nonzero fibers identified with $\mathfrak{t} \times \mathfrak{t}$, while the zero fiber is $T(\mathfrak{t}/W) \times_{\mathfrak{t}/W} S$. Taking the proper transform of Z'_n we get a subscheme whose fiber over zero is our Z_n.

Upshot: Z_n is the renormalized zero fiber of the union of graphs of $w \in W_n$.

By the way, if we don't restrict to the zero fiber, we get a version of affine Soergel bimodules which "controls" the category of monodromic sheaves on the total space of punctured determinant bundle of $\widetilde{\mathcal{F}\ell}$ (or, Koszul dually, loop rotation equivariant complexes on $\mathcal{F}\ell$). On the $Coh^Z(\mathfrak{t}/W)$ side this corresponds to deforming $Coh^G(\mathfrak{g})$ to Harish-Chandra bimodules on which one considers the Whittaker functor, see section 4 of my paper with Finkelberg:

https://arxiv.org/pdf/0707.3799.pdf