April 12: David Vogan (MIT), "Nilpotent orbits, K-types, and unitarity."

Chaoping Dong has been using the `atlas` software to investigate unitary representations of complex exceptional groups. He noticed that "non-unitary representations have more K-types with higher multiplicities." I will discuss how to make such a statement precise, and talk about an (entirely conjectural) way that one might try to prove it.

Here is part of the idea. Each irreducible Harish-Chandra module X has an *associated variety* $AV(X)$, which is a finite union of nilpotent orbits $\mathcal{O}_i \simeq K/K_i$ of K on $(\mathfrak{g}/\mathfrak{k})^*$. The associated variety invariant can be refined to attach a virtual algebraic representation $\xi_i(X) \in \widehat{K_i}$ to each orbit, and (roughly speaking)

$$X|_K \simeq \sum_i \mathrm{Ind}_{K_i}^K(\xi_i).$$

If X admits an invariant Hermitian form, then each virtual representation ξ_i decomposes as a sum $\xi_i = p_i + q_i$. The various p_i control the positive part of the signature of the form on X, and the q_i the negative part.

In this language, the easiest way for a representation to be non-unitary is that both p_i and q_i should be non-zero for some i; and the easiest way for a representation to be unitary is that the ξ_i should all have dimension 1 or zero.

Dong's observation can be interpreted as "the virtual representations tend to have dimension 1 for unitary X and dimension greater than one for non-unitary X." I will explain a way to try to prove such a statement, by relating it to the well-known fact, "an irreducible finite-dimensional representation of a noncompact simple Lie group is unitary if and only if it is one-dimensional."