September 5, 2012: David Vogan (MIT), *Realizing smooth representations.*

Joint work with David Jerison.

Gelfand's program of abstract harmonic analysis seeks to understand a manifold X with an action of a Lie group G in four steps:

(1) understand all irreducible unitary representations of G;
(2) construct the Hilbert space $L^2(X)$;
(3) decompose $L^2(X)$ as a "direct integral" of irreducible unitary representations.
(4) relate (hard) problems about X to $L^2(X)$, and so (via (3)) to (easier) problems about irreducible unitary representations.

We are interested in two (very old) technical problems arising in Gelfand's program. Suppose we wish to understand a G-invariant differential operator Δ_X. The first problem is that eigenfunctions of Δ_X may not be square-integrable, so relating them to $L^2(X)$ (step (4)) is subtle.

The second problem appears already in step (1), the construction of irreducible unitary representations. Natural candidates for such representations are simultaneous eigenspaces for G-invariant differential operators on X; but if (as very often happens) the eigenfunctions are not square-integrable, then it is not easy in this way to find natural Hilbert space representations. The general solution we propose is dualization: to replace *kernels* of differential operators acting on *distributions* by *cokernels* of differential operators acting on *compactly supported densities*.

I'll describe these problems and their resolution in some very familiar examples related to $SL(2,\mathbb{R})$ and one complex variable.