October 26: David Vogan (MIT), "Left cells and Harish-Chandra cells of Weyl group representations."

Suppose $G(R)$ is a real reductive algebraic group, and λ is a regular integral infinitesimal character. There is a finite set

$$B(\lambda) = \{X_1, \dots, X_n\}$$

of irreducible representations of infinitesimal character λ. Kazhdan-Lusztig theory provides a "W-graph" structure on $B(\lambda)$, which among other things provides a representation of W (defined over $\mathbb{Z}$) with basis $B(\lambda)$. The (directed) edges of the graph provide a preorder on $B(\lambda)$, telling when one representation can be obtained from another as a subquotient of a tensor product with a finite-dimensional representation. The equivalence classes for the preorder are called the "Harish-Chandra cells" for $B(\lambda)$. All the representations in a single Harish-Chandra cell have the same Gelfand-Kirillov dimension and associated variety, so it is useful to understand these cells.

Each Harish-Chandra cell inherits from $B(\lambda)$ a W-graph structure, and therefore constitutes a basis for a representation of W. The question I want to consider is this: how are Harish-Chandra cells related to the left cells defined by Kazhdan and Lusztig? This is the kind of question that the `atlas` software is able to study, and Birne Binegar proved in this way that for the real exceptional groups, every Harish-Chandra cell is isomorphic (as a Weyl group representation) to a Kazhdan-Lusztig left cell. For real forms of the classical Lie algebras, the same statement is true up to rank eleven; and McGovern has proved the same assertion for most of the "classical" classical groups (like $SO(2a+1, 2b)$).

Nevertheless (as McGovern also showed) there are Harish-Chandra cells that are *not* isomorphic to Kazhdan-Lusztig left cells. I will explain his example, and formulate some new problems in this (still mysterious) direction.