March 8: David Vogan (MIT), "Coxeter groups and regular polyhedra."

This is the seminar that in long-past years was called the "free pizza for graduate students" seminar. This year there is no free pizza for graduate students (who are instead invited to join the prospective new grad students for dinner at 6:30) but I *will* offer the traditional lack of new intellectual content.

Coxeter groups by definition admit a very simple set of generators and relations. The generating set is called S; each generator has order 2; and there are additional relations

$$(st)^{m(s,t)} = 1 \qquad (s \neq t \in S),$$

with $m(s,t)$ either an integer at least 2 or infinity. The relations can be encoded by a graph with vertex set S, in which s and t are joined by $m(s,t)-2$ edges. The corresponding Coxeter group is written $W(S)$.

A Coxeter group with two generators is a dihedral group, and therefore finite unless $m(s,t)$ is infinite. For most graphs with three or more vertices, this group is infinite. Coxeter gave a simple geometric realization of $W(S)$, a nice geometric criterion for $W(S)$ to be finite, and a complete list of all the cases in which $W(S)$ is finite. (Most of them are Weyl groups of compact Lie groups.)

What is slightly less well known is the relationship of Coxeter's classification to the classification of regular polyhedra in $\mathbb{R}^n$.

Theorem. *Suppose P is a regular polyhedron in $\mathbb{R}^n$. Then the isometry group of P is generated by n reflections*

$$S = \{s_1, \ldots, s_n\},$$

and is a Coxeter group $W(S)$. This defines a bijection between similarity classes of regular polyhedra, and finite Coxeter groups having a single-line graph

$$s_1 \ \overset{m(1,2)}{\overline{\qquad}} \ s_2 \ \overset{m(2,3)}{\overline{\qquad}} \ \cdots \ \overset{m(n-1,n)}{\overline{\qquad}} \ s_n$$

where all $m(i,i+1)$ are at least 3.

The "Schläfli symbol" of the regular polyhedron is $\{m(1,2), m(2,3), \ldots, m(n-1,n)\}$. I'll describe Coxeter's proof of this theorem.