Projective root systems, enhanced Dynkin diagrams and Weyl orbits

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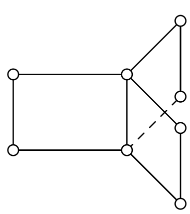
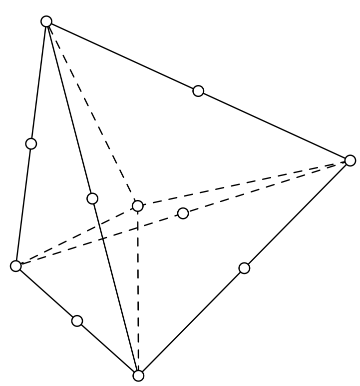
The talk is based on a joint paper with E. B. Dynkin which is now in progress. Our goal is to develop new tools for investigating classes $\mathcal{A}$ of conjugate semisimple subalgebras of semisimple Lie algebras, in particular, for a description of a natural partial order between such classes. (The relation $\mathcal{A}_1 \prec \mathcal{A}_2$ means that $A_1 \subset A_2$ for some $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$.)

One of tools is enhanced Dynkin diagrams which contain Dynkin diagrams as subdiagrams. The number of nodes in these diagrams is still not too big. For instance, it is equal to 8 for E_6, to 11 for E_7 and to 16 for E_8. (The diagrams for E_6, E_7 and E_8 are placed below. Nodes of the diagram for E_8 constitute 4×4 lattice on a torus.) Every node represents a pair of roots $(\alpha, -\alpha)$. We call these pairs projective roots. The Weyl group W acts on the set S of projective roots and therefore it acts on the space $\mathcal{V}$ of all subdiagrams of S isomorphic to Dynkin diagrams. Classes of conjugate regular subalgebras are in a 1-1 correspondence with Weyl orbits in $\mathcal{V}$.

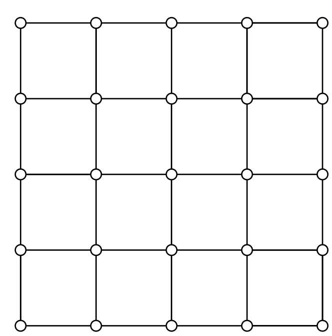
A special role is played by maximal orthogonal subsets M of S. All of them are conjugate (like Cartan subalgebras). We denote W^M and we call the core group the group of all elements of W preserving M. The elements of a Weyl orbit in $\mathcal{V}$ which are contained in M form an orbit of W^M. This allows to reduce problems related to classes of conjugate regular subalgebras to similar problems regarding to orbits of the core group. No reduction of this kind is possible without a transition from roots to projective roots.



E_6



E_7



E_8