Zeta-functions of plane curve singularities, Jacobian factors and DAHA

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The (uncolored) HOMFLY-PT polynomials $H(q,a)$, generalizing the Jones and other knot invariants of are defined as follows:

$$a^{1/2}H(\nwarrow\!\!\!\!\!\nearrow)-a^{-1/2}H(\nearrow\!\!\!\!\!\nwarrow)=(q^{1/2}-q^{-1/2})H(\uparrow\uparrow),\ \ H(\bigcirc)=1.$$

We will consider algebraic knots: intersections of unibranch plane curve singularities $0\in\mathcal{C}\subset\mathbb{C}^2$ with small $\mathbb{S}^3$ centered at 0. Let $\widetilde{H}$ be $a^{\bullet}q^{\bullet}H$ such that $\widetilde{H}\in\mathbb{Z}[q,a]$ and $\widetilde{H}(a=0)\in 1+q\mathbb{Z}[q]$. Their t-refinements, the Khovanov-Rozansky stable reduced polynomials $\widetilde{\mathcal{H}}(q,t,a)$, are significantly more involved. The connection is $\widetilde{\mathcal{H}}(q,q,a\mapsto -a)=\widetilde{H}(q,a)$.

Conjecturally, $\widetilde{\mathcal{H}}$ coincide with the geometric-motivic superpolynomials $\mathcal{H}_{\mathcal{C}}$, given in terms of the Jacobian factors $J_{\mathcal{C}}$ of $\mathcal{C}$ (Ch-Philipp). Moreover, conjecturally $\mathcal{H}_{\mathcal{C}}(q\mapsto qt,t,a)=L(q,t,a)\stackrel{\mathbf{def}}{=\!=}(1-t)Z(q,t,a)$ for the flagged generalization of the Galkin-Stöhr zeta-function Z of the ring of $\mathcal{C}$ over a finite field $\mathbb{F}_q$; $Z(q,t,a)$ are related to the so-called ORS polynomials. For any a-coefficient of $\mathcal{H}_{\mathcal{C}}(qt,t,a)$, the Riemann Hypothesis for t-zeros holds for sufficiently small q (a theorem); presumably at least for $0<q\leq 1/2$ at $a=0$ (so "distant" from $q=|\mathbb{F}|$).

Furthemore, conjecturally $\mathcal{H}_{\mathcal{C}}=\mathcal{H}^{daha}_{\mathcal{C}}$, where the latter are given in terms of the projective action of $PSL(2,\mathbb{Z})$ in DAHA; the chains of $\gamma\in PSL(2,\mathbb{Z})$ encode the topological types of $\mathcal{C}$. The DAHA-superpolynomials exist for any colores and torus iterated links (not only algebraic); this is the setting of arXiv:1709.07589v6. The latter contains a technique of recurrence relations for $\mathcal{H}^{daha}_{\mathcal{C}}$ in "families" of $\mathcal{C}$, which can be hopefully used to prove their coincidence with $\mathcal{H}_{\mathcal{C}}$.

From the physics perspective, (spherical) DAHA are closely related to CFT at $t=q$ and, generally, to q-VOA, q-W-algebras, and so on. On the other hand, the geometric-motivic direction is probably related to the Landau-Ginzburg Sigma Models for superpotentials W (equations of $\mathcal{C}$). The program in Vafa-Warner's paper "Catastrophes..." (1989) was to study Field Theories associated with LGSM as directly as possible in terms of W; motivic superpolynomials do exactly this.