Cohomological Hall Algebras of Curves & ADE singularities

Jt with
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I- Heuristics about Hall Algebras

$\mathcal{A}$ (nice) abelian category

$\mathcal{M}_{\mathcal{A}}$: moduli stack of objects in $\mathcal{A}$

$\rightsquigarrow$ Correspondence

$$\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \xleftarrow{q} \tilde{\mathcal{M}}_{\mathcal{A}} \xrightarrow{p} \mathcal{M}_{\mathcal{A}}$$

where $\tilde{\mathcal{M}}_{\mathcal{A}} = \{ 0 \to M' \to M \to M'' \to 0 \}$ "Hecke stack"

$$q : (0 \to M' \to M \to M'' \to 0) \mapsto (M'', M')$$

$$p : (0 \to M' \to M \to M'' \to 0) \mapsto (M)$$

$\rightsquigarrow$ · Fiber of p is a quot scheme
· q is a (truncated) Ext - bundle

insert what you want

For $X : \mathcal{M} \mapsto X(\mathcal{M})$ a [ ] theory with nice functorial properties,

You may try to define a (co)algebra structure on $X(\mathcal{M}_{\mathcal{A}}) =: H_{\mathcal{A}}$

$$* : X(\mathcal{M}_{\mathcal{A}}) \otimes X(\mathcal{M}_{\mathcal{A}}) \xrightarrow{p_* q^!} X(\mathcal{M}_{\mathcal{A}})$$

$$\Delta : X(\mathcal{M}_{\mathcal{A}}) \xrightarrow{q_* p^!} X(\mathcal{M}_{\mathcal{A}}) \otimes X(\mathcal{M}_{\mathcal{A}})$$

Examples

* $\mathcal{A}$ $\mathbb{F}_q$-linear, finite-dim $\mathrm{Ext}^{\cdot}$ spaces ("finitary") (Ringel)

$X(\mathcal{M}) = \mathrm{Fun}_0(\mathcal{M}(\mathbb{F}_q) \to \mathbb{C})$ (compact support)

Bonus: if $\mathrm{gldim}(\mathcal{A}) \le 1$ then $H_{\mathcal{A}}$ is a self-dual bialgebra "quantum group"

* $\mathcal{A}$ $\mathbb{C}$-linear, finite-dim $\mathrm{Ext}^{\cdot}$ spaces, $\mathrm{gldim}(\mathcal{A}) \le 1$

$\rightsquigarrow$ $\mathcal{M}_{\mathcal{A}}$ is smooth (Lusztig)

$X(\mathcal{M}) = D^b_{cst}(\mathcal{M}, \mathbb{C})$ (Laumon)

* $\mathcal{A}$ $\mathbb{C}$-linear, finite-dim $\mathrm{Ext}^{\cdot}$, $\mathrm{gldim}(\mathcal{A}) \le 2$ (S-V) (Sala-S) (Minets) (K-V) (Yu Zhao)

$X(\mathcal{M}) = H_*(\mathcal{M}, \mathbb{Q})$ (Borel-Moore homology)

may be replaced by Oriented B-M homology theory (K-theory, A., elliptic cohomology, ...) (Yang-Zhao)

sometimes (B. Davison) 3d $\to$ 2d reduction

D-T theory

* $\mathcal{A}$ CY3, $X(\mathcal{M}) = H_*(\mathcal{M})$ (various types) (Kontsevich Soibelman)
* $\mathcal{A}$ CY 2n, $X(\mathcal{M}) = H_*(\mathcal{M})$ (Joyce)

* ... (Porta-Sala: derived setup)

Personal Interest: Compute these algebras

1st historical example:

$\mathcal{A} = \{$abelian p-groups$\}$ (Steinitz)

$\mathcal{A} = \hat{\mathcal{O}}_x$-mod $x \in$ smooth curve $/\mathbb{F}_q$ (Macdonald)

$\mathcal{A} = \mathrm{Rep}^{nil}_{\mathbb{F}_q}(\cdot \circlearrowleft)$

$\rightsquigarrow$ $H_{\mathcal{A}} = \mathbb{C}[z_1, z_2, \dots]^{\mathfrak{S}_\infty} = \mathbb{C}[p_1, p_2, \dots]$

Macdonald ring of symmetric functions

A. Examples (Quivers)

$Q = (I, E)$ finite quiver. $\rightsquigarrow$ $\mathfrak{g}_Q$ Kac-Moody Lie algebra (Borcherds-(Bozec) Lie algebra)

- $\mathrm{Rep}\, Q$ gldim = 1
  $\rightsquigarrow$ "$\mathcal{A} \mapsto T^*\mathcal{A}$"
  $\mathrm{Rep}\, \Pi_Q$ gldim = 2 (CY2)
  Π_Q: preprojective algebra

(generated by $\{1_S, S$ simple nilp$\}$)

- $H^{sph}_{\mathbb{F}_q}(\mathrm{Rep}\, Q) = U_q^+(\mathfrak{g}_Q)$ (Ringel-Green)

- $H_{\mathbb{F}_q}(\mathrm{Rep}\, Q)$ "=" $U_q^+(\tilde{\mathfrak{g}}_Q))$ (Sevenhant - Van den Bergh, Bozec - S)

  where $\tilde{\mathfrak{g}}_Q$ is a certain <u>graded</u> Borcherds algebra

  s.t. $\tilde{\mathfrak{g}}_Q[0] = \mathfrak{g}_Q$

- Let $\Lambda_Q \subset \mathrm{Rep}\, \Pi_Q$ be the Lusztig (-Bozec) nilpotent variety (Lagrangian)

$$Y_\hbar^+(\mathfrak{g}_Q) \twoheadrightarrow H_*^{\mathbb{C}^*}(\Lambda_Q)$$ [S - Vasserot; Yang - Zhao]

($Y_\hbar^+$: Drinfeld new presentation; $\mathbb{C}^*$: dilat° along fibers of $\mathrm{Rep}\,\Pi_Q \to \mathrm{Rep}\, Q$)

Moreover $$H_*^{\mathbb{C}^*}(\Lambda_Q) \underset{v.s}{\simeq} U^+(\tilde{\mathfrak{g}}_Q[t]) \otimes \mathbb{C}[\hbar]$$

Expect:
- $\twoheadrightarrow$ is an isomorphism
- $\tilde{\mathfrak{g}}_Q = \mathfrak{g}_Q^{M-O} = \mathfrak{g}_Q^{BPS}$

  ($\mathfrak{g}_Q^{M-O}$: Maulik-Okounkov Lie algebra; $\mathfrak{g}_Q^{BPS}$: Davison-Meinhardt BPS Lie algebra)

Example: $Q = \cdot\circlearrowleft = \text{"}A_0^{(1)}\text{"}$

$$H^{sph}_{\mathbb{F}_q}(Q) = U^+(\mathbb{C})$$

$$H_{\mathbb{F}_q}(Q) = U_q^+(\widehat{gl(1)}) = Heis^+ \simeq \mathbb{C}_q[p_1, p_2, \dots]$$

$$H_*^{\mathbb{C}^*}(\Lambda_Q) = Y_\hbar^+(\widehat{gl(1)}) \quad \text{affine Yangian of } gl(1)$$

$$K^{\mathbb{C}^* \times \mathbb{C}^*}(\Lambda_Q) = U_q^+(\widehat{\widehat{gl(1)}}) \quad \text{Elliptic Hall algebra}$$

Note: · Λ_Q = moduli stack of 0-dim coh. sheaves on $\mathbb{C}^2$, supported on $\mathbb{C} \subset \mathbb{C}^2$.

· $SL(2,\mathbb{Z}) \rightsquigarrow U_q(\widehat{\widehat{gl(1)}})$ by algebra automorphisms

↰ "pointwise modificat° on a surface"

↪ Negut : $\mathbb{C}^2 \rightsquigarrow S$ smooth, projective

Y. Zhao

K-V

B. Examples (curves)

X: Smooth projective curve / k, connected, of genus g $\quad k \in \{\mathbb{F}_q, \mathbb{C}\}$

$k = \mathbb{F}_q$: $\quad H_{\mathbb{F}_q}(Coh(X)) \supset H^{sph}_{\mathbb{F}_q}(Coh(X))$ ← generated by $\{1_{Pic^\ell}, 1_{Coh_{0,d}}\}$

Both depend on X, via Weil numbers $\sigma_1, \dots, \sigma_{2g}$ (= Frob. eigenvalues in $H^1_{et}(\bar{X}, \bar{\mathbb{Q}}_\ell)$)

($g+1$ parameters)

Note: $Frob \in GSp(H^1_{et}(\bar{X}, \bar{\mathbb{Q}}_\ell))$

$\rightsquigarrow$ Integral form: $\quad R_g = K_0(Rep\, Gsp(2g, \mathbb{C})) \simeq \mathbb{C}[T_g]^{W_g}$ ← torus + Weyl grp of $GSp(2g, \mathbb{Q})$

For any $X/\mathbb{F}_q$ get a specialization map $\begin{cases} R_g \to \mathbb{C} \\ P \mapsto P(\sigma_1, \dots, \sigma_{2g}) \end{cases}$

Expect: $\exists\ \tilde{g}_g$ a certain Lie algebra in the category of $Gsp(2g, \mathbb{C})$-modules s.t.

· $H_{\mathbb{F}_q}(Coh(X)) \text{ "}\simeq\text{" } U^+_{R_g}(\tilde{g}_g)_{|Frob_X}$ (Hongjiy Yu)

· $H^{sph}_{\mathbb{F}_q}(Coh(X)) = U^+_{R_g}(\tilde{g}_g[0])_{|Frob_X}$

(H^{sph}: subalgebra gen by $\{1_{Pic^\ell}, 1_{Coh_{0,d}}\}$; $[0]$: trivial isotypic component)

Moreover,

$$\mathrm{Ch}(\tilde{\mathfrak{g}}_g) = \sum_{(r,d)} A_{g,r,d}\, e^{(r,d)}$$

where $A_{g,r,d}\,|_{\mathrm{Frob}_X} = \#\{\text{abs. indecomp. coh. sheaf on } X \text{ of rank } r \text{ \& deg } d\}/\sim$

Kac polynomial of genus g curves

. $T^* \mathrm{Coh}_X$ "$\simeq$" $\mathrm{Higgs}_X \supset \Lambda_X$ global nilpotent cone.

Expect

$$H_*^{\mathbb{C}^*}(\Lambda_X) \simeq Y_\hbar^+(\tilde{\mathfrak{g}}_g) \underset{\hbar\to 0}{\simeq} U^+(\tilde{\mathfrak{g}}_g[t])$$

$$H_{top}(\Lambda_X) \simeq U^+(\tilde{\mathfrak{g}}_g[0])$$

Rem: $Sp(2g,\mathbb{Z})$-action comes from action of the mapping class group

Examples

- $g=0$ $\quad H^{sph}_{\mathbb{F}_q}(X) = U_q^+(\hat{\mathfrak{sl}}_2)$ (Drinfeld half) (Kapranov) (Baumann-Kassel)

  $H_{\mathbb{F}_q}(X)$ "=" $U_q^+(\hat{\mathfrak{gl}}_2)$

- $g=1$ $\quad H^{sph}_{\mathbb{F}_q}(X) = U^+_{\sigma_1,\sigma_2}(\hat{\hat{\mathfrak{gl}}}_1) := EHA^+_{\sigma_1,\sigma_2}$ (Burban-S)

  $H_{\mathbb{F}_q}(X) = \bigotimes_{x\in|X|} U^+_{\sigma_1(x),\sigma_2(x)}(\hat{\hat{\mathfrak{gl}}}_1)$ (Fratila)

$\rightsquigarrow$ Expect: . $H_*^{\mathbb{C}^*}(\Lambda_{\mathbb{P}^1}) \simeq Y_\hbar^+(\hat{\mathfrak{gl}}_2)$ (Drinfeld[2] half)

(E elliptic curve) . $H_*^{\mathbb{C}^*}(\Lambda_E) \simeq Y_\hbar^+(\hat{\mathfrak{gl}}_1 \otimes H^*(X))$

triple loop algebra $\hat{\hat{\hat{\mathfrak{gl}}}}(1) \circlearrowleft SL(3,\mathbb{Z})$

"Pagoda algebra"

Mironov - Morozov - Zenkevich

II 2d Cohomological Hall algebras of ADE singularities

A. A_1-case

$\Gamma = \mathbb{Z}/_{2\mathbb{Z}} \curvearrowright \mathbb{C}^2$

$$S = \widetilde{\mathbb{C}^2/\!/\Gamma} \longrightarrow \mathbb{C}^2/\!/\Gamma \qquad \text{Fiber} = \mathbb{P}^1 = \mathcal{C}$$

$$\cong \quad T^*\mathbb{P}^1 \hookleftarrow \mathbb{P}^1 \quad \text{zero section}$$

$\mathcal{M}_S$ = moduli stack of coh. sheaves on S, supported on $\mathbb{P}^1 = \mathcal{C}$

$\simeq \Lambda_{\mathbb{P}^1}$ global nilpotent cone of $\mathbb{P}^1$

Notat°: $\quad Y_S^+ = \bigoplus_{r,d} H_*^{\mathbb{C}^*}(\mathcal{M}_{r,d})$

- Semistable COHAs: $\quad \ell \in \mathbb{Z} \quad \mathcal{M}^{ss}_{\mu=\ell}$ = substack of semistable Higgs sheaves of slope ℓ

$$\mathcal{M}^{ss}_{\mu=\ell} \hookrightarrow \bigsqcup_r \mathcal{M}_{r,\ell r} \qquad \text{(open)}$$

on $\mathbb{P}^1$, $(\mathcal{F},\varphi)$ semistable of slope $\ell \Rightarrow \begin{cases} \mathcal{F} \simeq \mathcal{O}(\ell)^{\oplus r} \\ \varphi = 0 \end{cases}$

$$\Rightarrow \mathcal{M}^{ss}_{\mu=\ell} \simeq \Lambda_Q \qquad Q = A_1$$

$$Y_S^+ \supset H_*^{\mathbb{C}^*}(\mathcal{M}_{\mu=\ell}) \twoheadrightarrow \underbrace{H_*^{\mathbb{C}^*}(\mathcal{M}^{ss}_{\mu=\ell})}_{\text{Semistable COHA}} \simeq Y^+(\mathfrak{sl}_2)$$

$\uparrow$ Drinfeld half

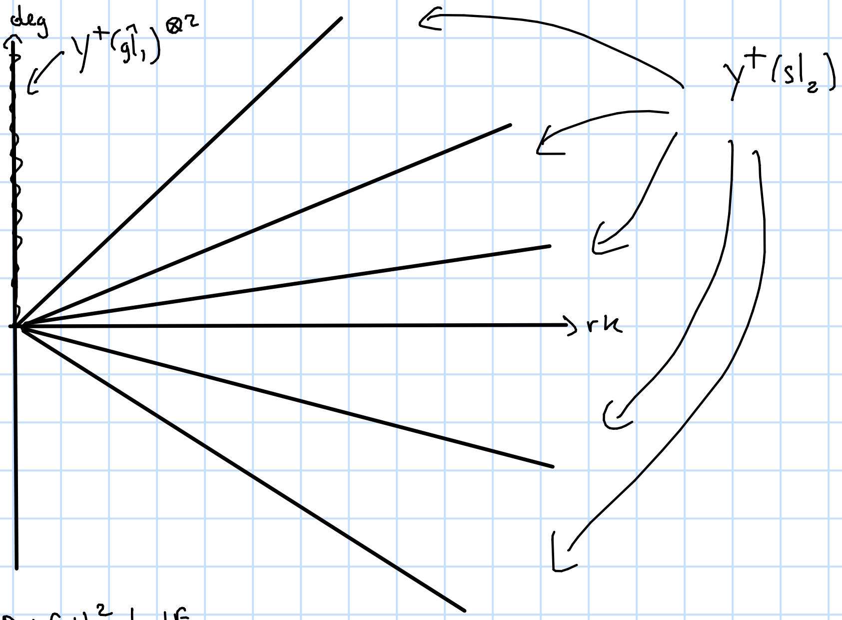
- $\mu = \infty \qquad \mathcal{M}_{0,d} \simeq$ moduli stack of 0-dim coh sheaves on $T^*\mathbb{P}^1$, supported on $\mathbb{P}^1$

$$H_*^{\mathbb{C}^*}(\mathcal{M}_{\mu=\infty}) = Y^+(\widehat{\mathfrak{gl}}_1) \otimes Y^+(\widehat{\mathfrak{gl}}_1[2])$$

$\uparrow$ coh. shift

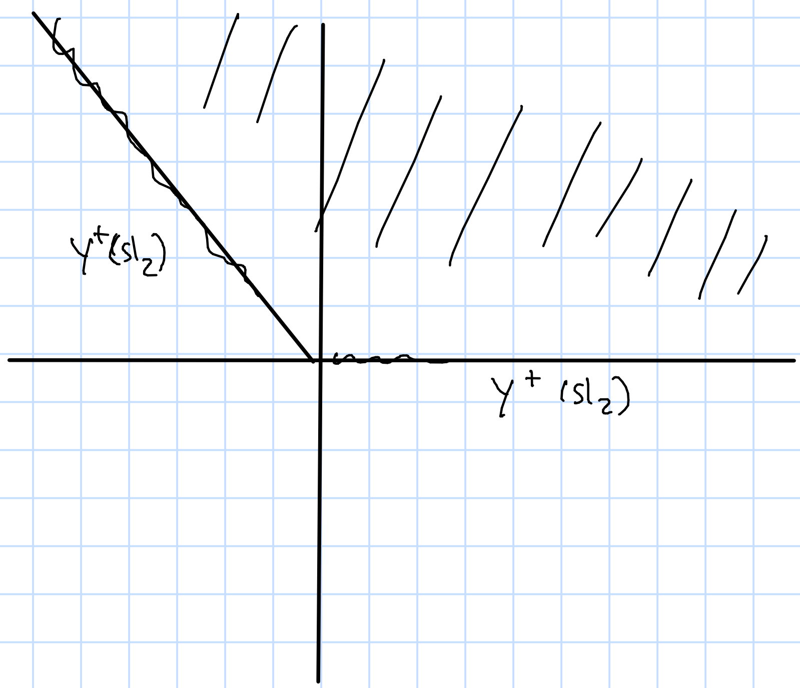
$$P_c(\mathbb{P}^1, u) = 1 + u^2$$

Picture



⊙ coh. degree

looks like $Y^+(\hat{sl}_2)$ Drinfeld[2] half



Thm (DS²V)

1) $\exists$ embedding $Y_S^+ \hookrightarrow Y_\hbar(\hat{sl}_2)^c$ completion

2) $Y_S^+ \simeq Y^+(\hat{gl}_1)^{\otimes 2} \rtimes Y^+(Lsl_2)$

3) PBW decomp

$$Y_S^+ \simeq Y(\hat{gl}_1)^{\otimes 2} \otimes \overrightarrow{\bigotimes_{\ell \in \mathbb{Z}}} Y^+(sl_2)$$

lift of semistable CoHA

$$Y^+(Lsl_2) = \langle T_{\omega_1}^{\ell}(e_n) \mid \ell \in \mathbb{Z}, n \in \mathbb{N} \rangle$$

For fixed ℓ, $\langle T_{\omega_1}^{\ell}(e_n) \mid n \in \mathbb{N} \rangle$ generate $Y^+(sl_2)$

Rems :

- $T^{\ell}_{-\omega}(e_n)$ is supported on infinitely many components of $\Lambda_{\mathbb{P}^1}$

ex: $$\sum_{\ell} [\mathrm{coh}_{1,\ell}] z^{\ell} = \Big(\sum_{\ell} T^{\ell}_{-\omega_1}(e_0)\cdot z^{\ell}\Big)\cdot\Big(\sum_{d\geq 0} [\mathrm{coh}_{0,d}] z^d\Big)$$

($[\mathrm{coh}_{1,\ell}]$ ↗ zero sect° of $\Lambda_{\mathbb{P}^1} \to \mathrm{Coh}_{\mathbb{P}^1}$)

- Morally Y^+_S & $Y^+(\hat{sl}_2)$ are two "halves" of $Y(\hat{sl}_2)$

$\rightsquigarrow$ a COHA version of Cramer's thm (for Hall algebras $/\mathbb{F}_q$)

(but we need completion)

- We can turn on $\mathbb{C}^* \curvearrowright \mathbb{P}^1$ $\rightsquigarrow$ 2-parameter version $Y_{\varepsilon_1,\varepsilon_2}(\hat{sl}_2)$

B - General ADE case

$\Gamma \subset SL_2(\mathbb{C})$ of type $Q^{(1)}$

$S = \widetilde{\mathbb{C}^2/\!/\Gamma} \longrightarrow \mathbb{C}^2/\!/\Gamma$ fiber $= \mathcal{C} = \bigsqcup_{i=1}^{N} \mathcal{C}_i$

$\mathcal{C}_i$ -2 rational curves in S, intersecting along Dynkin diagram of Q.

$\mathcal{M}_S = \mathrm{Coh}_{\mathcal{C}}(S) =$ moduli stack of coh. sheaves on S supported on $\mathcal{C}$.

$Y^+_S = H^{\mathbb{C}^*}_*(\mathcal{M}_S)$

Fix $\{D_i, i=1,\dots,N\}$ st $D_i\cdot\mathcal{C}_j = \delta_{ij}$

$\omega = \sum \xi_i D_i$ $\quad \xi_i > 0$ $\rightsquigarrow \mu_\omega(F) = \dfrac{\chi(F)}{\omega\cdot ch_1(F)}$

- $\mu = \infty$

$\mathrm{Coh}^{ss}_{\mu=\infty}$ = moduli stack of 0-dim sheaves on $\mathcal{C}$

$$H^{\mathbb{C}^*}_*(\mathrm{coh}^{ss}_{\mu=\infty}) \simeq Y(\hat{gl}_1) \otimes Y^+(\hat{gl}_1[z])^{\otimes N}$$

- $\mu = \xi_i^{-1}\ell$ $\quad \ell \in \mathbb{Z}$, $ch_1(F) = \mathcal{C}_i$ $\rightsquigarrow \bigoplus_r H^{\mathbb{C}^*}_*(\mathcal{M}^{ss}_{\mu=\xi_i^{-1}\ell,\ ch_1=r\mathcal{C}_i}) \simeq Y^+(sl_2)$

Thm (DS²V)

1) $\exists \; Y_S^+ \hookrightarrow Y_\hbar(\hat{\mathfrak{g}}_Q)^c$

2) $Y_S^+ \simeq Y^+(\hat{\mathfrak{gl}}(1))^{\otimes N+1} \rtimes Y_\hbar^{>}(L\mathfrak{g}_Q)$

3) Y_S^+ is gen by one copy of $Y^+_{T^*\mathbb{P}^1}$ for each $\mathcal{C}_i$,

modulo some relations (local, one for each $\mathcal{C}_i \cap \mathcal{C}_j$)

4) "PBW" $\quad Y_S^+ \underset{v.s}{\simeq} \overrightarrow{\bigotimes}_{Y^+(\hat{\mathfrak{gl}}_1)} Y^+_{T^*\mathbb{P}^1}$

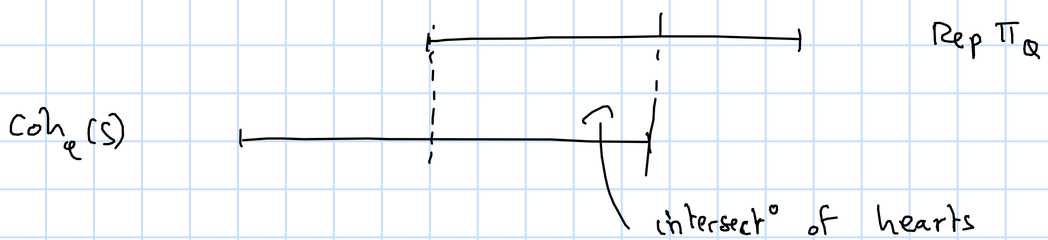
($\uparrow$ strange $\quad Y^+(\hat{\mathfrak{sl}}_n)$ built out of n ordered products of $\hat{\mathfrak{sl}}_2$!)

A few words about proofs

$$D^b(\mathrm{Coh}_\mathcal{C}(S)) \overset{\text{McKay}}{\simeq} D^b(\mathrm{Rep}\,\Pi_Q^{nil})$$

$$\mathcal{O}_{\mathcal{C}_i}(-1)[1] \longmapsto S_i \quad \text{simple}$$

$$\mathcal{O}_\mathcal{C} \longmapsto S_0$$



They are described in terms of slope stability

$$\mathcal{M}_S \hookleftarrow \mathcal{M}_S^{\geq 0} \simeq \Lambda_Q^{\leq 0} \hookrightarrow \Lambda_Q$$

Moreover, these correspond to torsion pair $\begin{cases}(\mathcal{T}_S, \mathcal{F}_S) \\ (\mathcal{T}_Q, \mathcal{F}_Q)\end{cases}$

$\rightsquigarrow \quad Y_S^+ = H_*(\mathcal{M}_S)$ acts on $H_*(\mathcal{M}_S^{\geq 0})$ on the right

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \wr\Vert$

$\quad\; Y_Q^+ = H_*(\Lambda_Q)$ —— $H_*(\Lambda_Q^{\leq 0})$ on the left

$\rightsquigarrow \quad Y_S^+ - Y_Q^+$ bimodule.

Use autoequivalences of $D^b(\mathrm{Rep}\,\Pi_Q)$ given by braid group to push this bimodule towards $\mathcal{M}_S$.

Prop: The equivalence $\mathrm{Aut}(D^b(\mathrm{Coh}_e(S))) \simeq \mathrm{Aut}(D^b(\mathrm{Rep}\,\Pi_Q^{nil}))$ sends $\cdot \otimes \mathcal{O}(-\Sigma D_i)$ to $T_{-\rho^\vee} \in \hat{B}^e_Q$

↖ extended affine braid

$$\mathrm{Coh}_e(S) \text{ "}\simeq\text{" } \lim_{\ell\to\infty} T^\ell_{-\rho^\vee}(\mathrm{Rep}\,\Pi_Q^{\leq 0})$$

$$\rightsquigarrow \quad H_*^{e*}(\mathcal{M}_S) = \varprojlim H_*(\mathrm{Rep}\,\Pi_Q^{\leq 0})$$

To understand this projective limit, we only need to understand compatibility between $\hat{B}^e_Q$ & $Y^+(\Lambda_Q)$

$\rightsquigarrow$ get reduced to T_i

$$\begin{cases} T_i(S_i) = S_i[1] \\ j \neq i \quad T_i(S_j) \in \mathrm{Rep}(\Pi_Q) \end{cases}$$

$$\Lambda^i = \{ M \in \mathrm{Rep}\,\Pi_Q \mid \mathrm{Hom}(S_i, M) = \{0\} \} \hookrightarrow \Lambda$$

$${}^i\Lambda = \{ \text{———} \mid \mathrm{Hom}(M, S_i) = \{0\} \} \hookrightarrow \Lambda$$

Prop

$$\begin{array}{ccc} H_*(\Lambda^i_Q) & \xrightarrow[\sim]{T_i} & H_*({}^i\Lambda_Q) \\ \wr\| & \circlearrowright & \wr\| \\ Y_h^+(\mathfrak{g}_Q)/Y_h^+ Y_i^+ & \xrightarrow{\overline{T}_i} & Y_h^+(\mathfrak{g}_Q)/Y_h^+ Y_i^+ \end{array}$$

↑ Kodera's braid group act° on affine Yangians

Best proof: (?) $\hat{B}^e_Q$ acts on Maulik–Okounkov Yangian via BGP reflect° functors. It is given by Kodera's operators

Our proof: Kirwan surjectivity + Lusztig thm on $\hat{B}^e_Q$ & canonical basis