Analytic version of the Langlands correspondence for complex curves and quantum integrable systems

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Introduction

I will talk about a joint work with Pavel Etingof and David Kazhdan:

P. Etingof, E. Frenkel, and D. Kazhdan,

An analytic version of the Langlands correspondence for complex curves, **arXiv:1908.09677**, to appear in the Dubrovin Memorial Volume published by AMS;

Hecke operators and analytic Langlands correspondence for curves over local fields, **arXiv:2103.01509**;

and another paper in preparation.

Answering a question posed by R.P. Langlands, we propose an analytic version of the Langlands correspondence for complex curves. Along the way, we obtain interesting quantum integrable systems.

The *unramified Langlands correspondence* for a curve $X/\mathbb{F}_q$ and a connected reductive algebraic group G split over $\mathbb{F}_q$: $F = \mathbb{F}_q(X)$

On one side, the set $\mathrm{Bun}_G(\mathbb{F}_q)$ of isomorphism classes of principal G-bundles on X. If G is *simple*, then for $x \in X(\mathbb{F}_q)$,

$$\mathrm{Bun}_G(\mathbb{F}_q) \simeq G(\mathbb{F}_q[X\backslash x])\backslash G(\mathbb{F}_q((t_x)))/G(\mathbb{F}_q[[t_x]])$$

This is a discrete countable set with a natural measure assigning $[\mathcal{P}] \mapsto 1/|\operatorname{Aut}(\mathcal{P})(\mathbb{F}_q)|$ (well-defined because $\operatorname{Aut}(\mathcal{P})(\mathbb{F}_q)$ is finite).

Use this measure to define a Hermitean inner product on $\mathbb{C}$-valued functions on $\mathrm{Bun}_G(\mathbb{F}_q) \longrightarrow$ Hilbert space $\mathcal{H}_G$ of L^2 functions.

One side: Joint spectrum of the commuting Hecke operators acting on $\mathcal{H}_G$ (these are labeled by $x \in |X|$, $\lambda \in \check{P}^+$).

Other side: Galois data associated to X and the Langlands dual group ${}^LG$ (essentially, $\mathrm{Gal}(\overline{F}/F) \to {}^LG$).

Now suppose that X is a (smooth projective) curve over $\mathbb{C}$.

In this case, the Langlands correspondence has been traditionally formulated in terms of sheaves rather than functions. It is usually referred to as *geometric* or *categorical*.

Instead of functions on Bun_G, one considers the derived category of D-modules on Bun_G, and instead of Hecke operators one considers Hecke functors on this category.

It turns out that there is a *function-theoretic* (or *analytic*) version for complex curves as well. The two versions complement each other.

Analogy: correlation functions in 2D conformal field theory are single-valued bilinear combinations of (multi-valued) conformal and anti-conformal blocks.

Namely, it is possible to associate to Bun_G of $X/\mathbb{C}$ (and more generally X/F, where F is a local field) a natural Hilbert space $\mathcal{H}_G$ and define analogues of the Hecke operators acting on a dense subspace of $\mathcal{H}_G$. We conjecture that they give rise to mutually commuting normal compact operators on $\mathcal{H}_G$.

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on Bun_G introduced by Beilinson and Drinfeld, as well as their complex conjugates.

We conjecture that the joint spectrum of this commutative algebra (properly understood) can be identified with the set of ${}^L G$-opers on X whose monodromy is in the *split real form* of ${}^L G$, up to conjugation (these play the role of the Galois data).

This statement may be viewed as an analytic Langlands correspondence for complex curves.

The spectral problem may be viewed as a quantum integrable system.

Basic definitions

X – smooth projective irreducible curve over $\mathbb{C}$

$S \subset X(\mathbb{C})$ – finite subset

K_X – canonical line bundle on X

G – connected simple algebraic group over $\mathbb{C}$

${}^L G$ – the Langlands dual group

$\mathrm{Bun}_G = \mathrm{Bun}_G(X, S)$ – algebraic stack of pairs $(\mathcal{F}, r_S)$, where $\mathcal{F}$ is a G-bundle on X and r_S is a B-reduction of $\mathcal{F}|_S$

$\mathrm{Bun}_G^{\mathrm{rs}} = \mathrm{Bun}_G^{\mathrm{rs}}(X, S) \subset \mathrm{Bun}_G(X, S)$ – substack of those stable pairs $(\mathcal{F}, r_S)$ whose group of automorphisms is equal to the center $Z(G)$ of G

Assumption:

$\mathrm{Bun}^{\mathrm{rs}}_G(X,S)$ is *open and dense* in $\mathrm{Bun}_G(X,S)$, i.e. one of the following cases:

1. the genus of X is greater than 1, and S is arbitrary;
2. X is an elliptic curve and $|S| \geq 1$;
3. $X = \mathbb{P}^1$ and $|S| \geq 3$.

The stack $\mathrm{Bun}^{\mathrm{rs}}_G(X,S)$ is a $Z(G)$-gerbe over a smooth algebraic variety $Bun^{\mathrm{rs}}_G(X,S)$ (coarse moduli space).

For our purposes, $Bun^{\mathrm{rs}}_G(X,S)$ is a good replacement for $\mathrm{Bun}^{\mathrm{rs}}_G(X,S)$ because all objects we need descend to $Bun^{\mathrm{rs}}_G(X,S)$.

Hilbert space

K_{Bun} – the canonical line bundle on Bun_G

For simply-connected G, Beilinson and Drinfeld have constructed a square root $K_{\mathrm{Bun}}^{1/2}$ of K_{Bun}. For a general G, their construction sometimes requires a choice of a square root of the canonical line bundle K_X on X. If so, we will make such a choice (however, the line bundle $\Omega_{\mathrm{Bun}}^{1/2}$ below does not depend on this choice).

We'll use the same notation for the restriction of this $K_{\mathrm{Bun}}^{1/2}$ to Bun_G^{rs}.

Given a holomorphic line bundle $\mathcal{L}$ on a variety Y, let

$|\mathcal{L}| := \mathcal{L} \otimes \overline{\mathcal{L}}$ be the corresponding C^∞ line bundle.

Set $\Omega_{\mathrm{Bun}}^{1/2} := |K_{\mathrm{Bun}}^{1/2}|$ – the line bundle of half-densities on Bun_G^{rs}.

Hilbert space

Let V_G – space of smooth compactly supported sections of $\Omega^{1/2}_{\mathrm{Bun}}$ over Bun_G^{rs}, and let

$\langle \cdot, \cdot \rangle$ – positive-definite Hermitian form on V_G given by

$$\langle v, w \rangle := \int_{Bun_G^{\mathrm{rs}}} v \cdot \overline{w}, \qquad v, w \in V_G$$

$\mathcal{H}_G$ – the Hilbert space completion of V_G

What kind of operators could act on the Hilbert space $\mathcal{H}_G$?

1. holomorphic differential operators;
2. anti-holomorphic differential operators;
3. Hecke (integral) operators.

Challenges: Differential operators are unbounded. It is a highly non-trivial task to define their self-adjoint (or normal) extensions, which is necessary to be able to make sense of the notion of their joint spectra on $\mathcal{H}_G$ (and there could be different choices).

Hecke operators are also initially defined on a dense subspace of $\mathcal{H}_G$. But we conjecture that they extend by continuity to normal compact operators on the entire $\mathcal{H}_G$. If one proves this, one gets a good spectral problem for both Hecke & differential operators since one can show that they commute (in the sense we'll discuss later).

Holomorphic differential operators

Consider the case of simply-connected G and $|S| = \emptyset$ (so $g > 1$).

Let $\mathcal{D}_G$ be the sheaf of algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\mathrm{Bun}}^{1/2}$ on Bun_G.

$$D_G := \Gamma(\mathrm{Bun}_G, \mathcal{D}_G)$$

Theorem 1 (Beilinson & Drinfeld)

$D_G \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(X)$, *where* $\mathrm{Op}_{^LG}(X)$ *– space of* ^LG*-opers on* X*.*

Definition. An ^LG-oper on a curve X is a holomorphic ^LG-*bundle* with a holomorphic *connection* ∇ and a *reduction to a Borel subgroup* ^LB which is in a special relative position with ∇.

Example (to be discussed later). A PGL_2-oper on X is a *projective connection*, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z)$: $K_X^{-1/2} \to K_X^{3/2}$.

Beilinson and Drinfeld derived their theorem from a **local result**:

Fix $x \in X$, and let $F_x \simeq \mathbb{C}((t))$ – completion of $F = \mathbb{C}(X)$

$\mathfrak{g}$ – simple Lie algebra, and $\widehat{\mathfrak{g}}_x$ – affine Kac–Moody algebra:

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_x \to \mathfrak{g} \otimes F_x \to 0$$

$$[A \otimes f, B \otimes g] = [A, B] \otimes fg - (A,\ B) \cdot \mathrm{Res}_x\, f dg \cdot \mathbf{1}$$

For $k \in \mathbb{C}$, let $\widetilde{U}_k(\widehat{\mathfrak{g}}_x)$ be the *completion* of $U(\widehat{\mathfrak{g}}_x)/(\mathbf{1} - k)$.

This is the completed enveloping algebra of $\widehat{\mathfrak{g}}_x$ at level k.

Theorem 2 (Victor Kac)

If $k \neq -h^\vee$, the center of $\widetilde{U}_k(\widehat{\mathfrak{g}}_x)$ is trivial.

Now let $Z(\widehat{\mathfrak{g}}_x)$ be the center of $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}}_x)$ (critical level).

Theorem 3 (Boris Feigin & E.F.)

$$Z(\widehat{\mathfrak{g}}_x) \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(D_x^\times)$$

$\mathrm{Op}_{^LG}(D_x^\times)$ – *the space of* ${}^LG$*-opers on* $D_x^\times := \mathrm{Spec}\, F_x$.

This isomorphism satisfies various compatibilities that make it unique up to an automorphism of the Dynkin diagram of $\mathfrak{g}$.

The theorem is derived from its vertex algebra version:

Let $Z(V_{-h^\vee}(\mathfrak{g})$ be the center of the vertex algebra $V_{-h^\vee}(\mathfrak{g})$. Then

$$Z(V_{-h^\vee}(\mathfrak{g})) \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(D_x), \qquad D_x = \mathrm{Spec}\,\mathcal{O}_x$$

Here $\mathcal{O}_x \simeq \mathbb{C}[[t]], \qquad F_x \simeq \mathbb{C}((t))$

Example. Let $G = SL_2, {}^LG = PGL_2$. A *PGL_2-oper* on $D_x^\times$ is the same as a *projective connection*, i.e. a second-order holomorphic differential operator of the form $\partial_t^2 - v(t)$: $K_X^{-1/2} \to K_X^{3/2}$

$$v(t) = \sum_{n \in \mathbb{Z}} v_n t^{-n-2}$$

$$\mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x^\times) \simeq \varprojlim \mathbb{C}[v_n]_{n \in \mathbb{Z}} / (v_m)_{m>N}$$

$$Z(\widehat{\mathfrak{sl}}_{2,x}) \simeq \varprojlim \mathbb{C}[S_n]_{n \in \mathbb{Z}} / (S_m)_{m>N}$$

where S_n are the Sugawara operators.

Isomorphism $Z(\widehat{\mathfrak{sl}}_{2,x}) \simeq \mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x^\times)$ sends $S_n \mapsto v_n$

From local to global

$$\mathrm{Bun}_G \simeq G(\mathbb{C}[X\backslash x])\backslash G(F_x)/G(\mathcal{O}_x)$$

$\widehat{\mathfrak{g}}_x$ acts on sections of a $G(\mathcal{O}_x)$-equivariant line bundle on $G(X\backslash x)\backslash G(F_x)$, which descends to a square root $K^{1/2}$ of the canonical line bundle on Bun_G. Central element $\mathbf{1} \mapsto -h^\vee$.

Hence $Z(\widehat{\mathfrak{g}}_x) \to D_G$, algebra of global hol. diff. operators on $K^{1/2}$.

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} Z(\widehat{\mathfrak{g}}_x) & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D_x^\times) \\ \downarrow & & \downarrow \\ D_G & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(X) \end{array}$$

Anti-holomorphic differential operators

Complex conjugates of elements of D_G are global anti-holomorphic differential operators acting on $\overline{K}^{1/2}_{\mathrm{Bun}}$.

They generate a commutative algebra $\overline{D}_G$.

$\overline{D}_G \simeq \mathrm{Fun}\,\overline{\mathrm{Op}}_{^LG}(X)$

$\mathcal{A}_G := D_G \otimes \overline{D}_G$ is a commutative algebra acting on C^∞ sections of the line bundle $\Omega^{1/2}_{\mathrm{Bun}} = K^{1/2}_{\mathrm{Bun}} \otimes \overline{K}^{1/2}_{\mathrm{Bun}}$ on Bun^{rs}_G.

Let $\widetilde{V}_G$ be the space of smooth sections of $\Omega^{1/2}_{\mathrm{Bun}}$ on $Bun^{\mathrm{vs}}_G \subset Bun^{\mathrm{rs}}_G$, the moduli space of *very stable G-bundles* (i.e. those $\mathcal{F}$ which do not admit non-zero $\phi \in \Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes K_X)$ taking nilpotent values everywhere).

"Doubling" of the quantum Hitchin system

Given a homomorphism $\Lambda : \mathcal{A}_G \to \mathbb{C}$, denote by $\widetilde{V}_{G,\Lambda}$ the corresponding eigenspace of $\mathcal{A}_G$ in $\widetilde{V}_G$.

$\Lambda = (\chi, \mu)$, where $\chi \in \mathrm{Op}_{^LG}(X)$, $\mu \in \overline{\mathrm{Op}}_{^LG}(X)$.

If f is a non-zero element of $\widetilde{V}_{G,(\chi,\mu)}$, then it satisfies two systems of differential equations:

(1) $P \cdot f = \chi(P) f, \qquad P \in D_G$

(2) $Q \cdot f = \mu(Q) f, \qquad Q \in \overline{D}_G$

$\chi(P_1 P_2) = \chi(P_1)\chi(P_2)$

$\chi : D_G \to \mathbb{C}$

System (1) is known as the *quantum Hitchin system*.

System (2) is its anti-holomorphic analogue.

$\chi \in \mathrm{Spec}\, D_G = \mathrm{Op}_{^LG}(X)$

The corresponding left $\mathcal{D}_G$-module

$$\Delta_\chi := \mathcal{D}_G \underset{D_G}{\otimes} \mathbb{C}_\chi$$

was introduced and studied by Beilinson and Drinfeld, who have proved that Δ_χ is a Hecke eigensheaf corresponding to the ${}^L G$-oper χ under the geometric/categorical Langlands correspondence.

Moreover, they have shown that the restriction of Δ_χ to Bun_G^{vs} is a vector bundle with a projectively flat connection (of a rank that grows exponentially with the genus of X).

Local sections of Δ_χ over Bun_G^{vs} are local holomorphic solutions of system (1). They are *multi-valued* and the monodromy is rather complicated, which is why there is no natural way in general to attach to a given χ a specific holomorphic half-form. (Even if there were single-valued solutions, it wouldn't be clear which one to choose.) Instead, we attach a whole $\mathcal{D}_G$-module on Bun_G to χ.

Likewise, to $\mu \in \overline{\mathrm{Op}}_{^LG}(X)$ we attach an anti-holomorphic D-module $\overline{\Delta}_\mu$ whose local sections on Bun_G^{vs} are local anti-holomorphic solutions of system (2), also multi-valued.

However, if we look for *smooth solutions* of systems (1) and (2) simultaneously, it is possible that for some χ and μ there will be a single-valued solution, which can be written locally in bilinear form

$f = \sum_{i,j} a_{ij} \; \phi_i(\mathbf{z})\overline{\psi}_j(\overline{\mathbf{z}})$

$\{\phi_i\}$ – local sections of Δ_χ

$\{\overline{\psi}_j\}$ – local sections of $\overline{\Delta}_\mu$.

This actually implies that $\dim \widetilde{V}_{G,(\chi,\mu)} < \infty$.

Moreover, if Δ_χ is irreducible and has regular singularities (for $G = SL_n$, this follows from the results of Dennis Gaitsgory) and $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\dim \widetilde{V}_{G,(\chi,\mu)} = 1$.

Conjecture 4

1. All $\widetilde{V}_{G,(\chi,\mu)} \subset \mathcal{H}_G$
2. There is an orthogonal decomposition

   $$\mathcal{H}_G = \widehat{\bigoplus}_{(\chi,\mu)} \widetilde{V}_{G,(\chi,\mu)}$$

3. If $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\mu = \tau(\overline{\chi})$, where τ is the Chevalley involution on ${}^L G$ and $\chi \in \mathrm{Op}_{^LG}(X)_{\mathbb{R}}$.

Definition. $\mathrm{Op}_{^LG}(X)_{\mathbb{R}}$ is the set of ${}^L G$-opers on X such that the *monodromy representation* $\rho_\chi : \pi_1(X, p_0) \to {}^L G(\mathbb{C})$ is isomorphic to its complex conjugate, i.e. $\rho_\chi \simeq \overline{\rho}_\chi$.

We expect that $\mathrm{Op}_{^LG}(X)_{\mathbb{R}}$ is a *discrete subset* of $\mathrm{Op}_{^LG}(X)$. This is known for ${}^L G = PGL_2$ (G. Faltings).

For $G = PGL_2$, Conjecture 4 implements ideas of J. Teschner.

We expect that $\mathrm{Op}_{^LG}(X)_{\mathbb{R}}$ coincides with the set of all ^LG-opers on X with *real monodromy*, i.e. such that the image in $^LG(\mathbb{C})$ of the monodromy representation

$$\rho_\chi : \pi_1(X, p_0) \to {}^LG$$

associated to χ is contained, up to conjugation, in the split real form $^LG(\mathbb{R})$ of $^LG(\mathbb{C})$.

This is known for $G = PGL_2$ and we can prove it for general G in the case when there is at least one point with Borel reduction (i.e. $|S| \neq \emptyset$).

Quantum integrable system

In some cases, the global differential operators (and the Hecke operators) can be written down explicitly, and then one obtains interesting quantum integrable systems. Our results and conjectures give a description of the spectra of the quantum Hamiltonians in these models.

Specifically, consider the case of $X = \mathbb{P}^1$ and

$$S = \{z_1, \ldots, z_N, \infty\}$$

Then the corresponding quantum integrable system is a **double** of the *Gaudin model* combining both holomorphic and anti-holomorphic degrees of freedom.

Let $G = SL_2$. Then the moduli space $Bun^{\rm rs}_{SL_2}$ is an open dense subspace of

$$(\mathbb{P}^1)^{N+1}/SL_2^{\rm diag} = (\mathbb{P}^1)^N/B^{\rm diag}$$

↑ ith copy x_i

$e^{(i)} \mapsto \partial_{x_i}$
$h^{(i)} \mapsto -2x_i\partial_{x_i}$
$f^{(i)} \mapsto -x_i^2\partial_{x_i}$

We have the Gaudin operators

$$H_i = \sum_{j \neq i} \frac{e^{(i)} \otimes f^{(j)} + f^{(i)} \otimes e^{(j)} + \frac{1}{2} h^{(i)} \otimes h^{(j)}}{z_i - z_j}, \quad i = 1, \ldots, N$$

which commute with the diagonal action of SL_2. They give rise holomorphic differential operators on $Bun^{\rm rs}_{SL_2}$.

(hol.)

In the past, looked at their action on the space of global sections of the line bundle $\boxtimes_{i=1}^N \mathcal{L}_{\lambda_i} \boxtimes \mathcal{L}_{\lambda_\infty}$, which is $(\otimes_{i=1}^N V_{\lambda_i} \otimes V_{\lambda_\infty})^{SL_2}$.

The joint eigenvalues of the H_i correspond to PGL_2-opers with regular singularities at $z_1, \ldots, z_N, \infty$ and *trivial monodromy*.

Now we look instead at the Hilbert space $\mathcal{H}$, which is the space of L^2 sections of the line bundle $\boxtimes_{i=1}^N |\mathcal{L}_{-1}| \boxtimes |\mathcal{L}_{-1}|$ of half-densities on $(\mathbb{P}^1)^{N+1}/SL_2^{\text{diag}}$.

It carries an action of the Gaudin Hamiltonians $H_i, i = 1, \ldots, N$ and their anti-holomorphic analogues $\overline{H}_i, i = 1, \ldots, N$.

The algebra $\mathcal{A}_{\mathbb{R}} = \mathbb{C}[H_i + \overline{H}_i, (H_i - \overline{H}_i)/i]_{i=1,\ldots,N}$ has a self-adjoint extension.

It turns out that if $\{\mu_i\}$ are the joint eigenvalues of $H_i, i = 1, \ldots, N$, then the second order Fuchsian differential operator on $\mathbb{P}^1$

$$\frac{\partial^2}{\partial z^2} + \sum_{i=1}^N \frac{1}{4(z - z_i)^2} - \sum_{i=1}^N \frac{\mu_i}{z - z_i}$$

has *real monodromy* representation $\pi_1(\mathbb{P}^1 \backslash S) \to PGL_2(\mathbb{R})$.

Moreover, there is a bijection between the spectra of the self-adjoint extension of $\mathcal{A}_{\mathbb{R}}$, and such Fuchsian operators.

Hecke operators

Proving Conjecture 4 directly is a daunting task. This is where the third set of operators on $\mathcal{H}_G$ – integral Hecke operators – comes in handy.

Though they are also initially defined on a dense subspace of $\mathcal{H}_G$ (like diff. operators), we conjecture that, unlike the differential operators, they extend to (mutually commuting) continuous operators on the entire $\mathcal{H}_G$, which are moreover *normal* and *compact* with trivial common kernel.

If so, then by a general result of functional analysis, $\mathcal{H}_G$ decomposes into a (completed) direct sum of mutually orthogonal *finite-dimensional eigenspaces* of the Hecke operators. Moreover, we can show that they commute with the differential operators, and so the Compactness Conjecture can be used to prove Conjecture 4.

In fact, Hecke operators can be defined for curves over any local field.

For non-archimedian local fields, these operators were essentially defined by A. Braverman and D. Kazhdan in *Some examples of Hecke algebras for two-dimensional local fields*, Nagoya Math. J. Volume 184 (2006), 57-84.

For $G = PGL_2, X = \mathbb{P}^1$, Hecke operators were studied by M. Kontsevich in his paper *Notes on motives in finite characteristic* (2007). In his letters to us (2019) he conjectured compactness of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over $\mathbb{C}$ could be used to construct an analogue of the Langlands correspondence was suggested in 2018 by R.P. Langlands, who attempted to construct them in the case when $G = GL_2$, X is an elliptic curve, and $S = \emptyset$ (however, for an elliptic curve X we can only define Hecke operators if $|S| \neq \emptyset$).

For a dominant coweight λ of G, denote by

$$q : Z(\lambda) \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X$$

the *Hecke correspondence* attached to λ. Let

$$p_{1,2} : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to \mathrm{Bun}_G, \quad p_3 : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to X$$

be the projections, and set $q_i := p_i \circ q$.

The following is due to Beilinson–Drinfeld and Braverman–Kazhdan.

Theorem 5

There exists an isomorphism

$$a : q_1^*(K_{\mathrm{Bun}}^{1/2}) \simeq q_2^*(K_{\mathrm{Bun}}^{1/2}) \otimes \omega_2 \otimes q_3^*(K_X^{-\langle\lambda,\rho\rangle})$$

where ω_2 is the relative canonical bundle along the fibers of $q_2 \times q_3$ and ρ is the half sum of positive roots.

The isomorphism a gives rise to an isomorphism

$$|a| : q_1^*(\Omega_{\mathrm{Bun}}^{1/2}) \simeq q_2^*(\Omega_{\mathrm{Bun}}^{1/2}) \otimes \Omega_2 \otimes q_3^*(|K_X|^{-\langle\lambda,\rho\rangle})$$

where $\Omega_2 := |\omega_2|$ is the relative line bundle of densities along the fibers of $q_2 \times q_3$. Let

$$U_G(\lambda) := \{\mathcal{F} \in Bun_G^{\mathrm{rs}} | (q_2(q_1^{-1}(\mathcal{F})) \subset Bun_G^{\mathrm{rs}}\}$$

This is an open subset of Bun_G^{rs}, which is dense if

$$\dim \mathrm{Bun}_G = \dim G \cdot (g-1) + \dim G/B \cdot |S| \quad (g > 1)$$

is sufficiently large. (For example, for $G = PGL_2, \lambda = \omega_1$, this is so if $\dim \mathrm{Bun}_G > 1$.)

Assume that $U_G(\lambda) \subset Bun_G^{\mathrm{rs}}$ is dense and let $V_G(\lambda) \subset V_G$ be the subspace of half-densities f such that $\mathrm{supp}(f) \subset U_G(\lambda)$.

$$Z_{\mathcal{G},x} := (q_2 \times q_3)^{-1}(\mathcal{G} \times x), \qquad \mathcal{G} \in \mathrm{Bun}_G(\mathbb{C}), \quad x \in X(\mathbb{C})$$

It is compact and isomorphic to the closure $\overline{\mathrm{Gr}}_\lambda$ of the $G[[z]]$-orbit Gr_λ in the affine Grassmannian of G.

The results of Braverman–Kazhdan imply that for any $f \in V_G(\lambda)$ and $x \in X(\mathbb{C})$, the restriction of the pull-back $q_1^*(f)$ to $Z_{\mathcal{G},x}$ is a well-defined measure with values in the line $|\Omega_{\mathrm{Bun}}|_{\mathcal{G}}^{1/2} \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$.

Hence for any $f \in V_G(\lambda)$, the integral

$$(\widehat{H}_\lambda(x) \cdot f)(\mathcal{G}) := \int_{Z_{\mathcal{G}}^x(F)} q_1^*(f)$$

is absolutely convergent for all $\mathcal{G} \in Bun_G^{\mathrm{rs}}(\mathbb{C})$ and belongs to the space V_G of compactly supported smooth sections on $Bun_G^{\mathrm{rs}}(\mathbb{C})$.

Therefore this integral defines a *Hecke operator*

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to V_G \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$$

Thus, we obtain an operator

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to \mathcal{H}_G \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$$

Conjecture 6 (Compactness Conjecture)

1. For any identification $(K_X^{1/2})_x \cong \mathbb{C}$, the corresponding operators $V_G(\lambda) \to \mathcal{H}_G$ extend to a family of commuting compact normal operators on $\mathcal{H}_G$, which we denote by $H_\lambda(x)$.
2. $H_\lambda(x)^\dagger = H_{-w_0(\lambda)}(x)$.
3. $\bigcap_{\lambda,x} \mathrm{Ker} H_\lambda(x) = \{0\}$.

Remark. We expect that integrals defining Hecke operators $H_\lambda(x)$ are absolutely convergent for all $f \in V_G$.

From now on we **assume** that Compactness Conjecture holds.

Let $\mathbb{H}_G$ be the commutative algebra generated by operators $H_\lambda(x), \lambda \in \check{P}^+, x \in X$. Denote by $\mathrm{Spec}(\mathbb{H}_G)$ its spectrum.

Corollary 7

There is an orthogonal decomposition

$$\mathcal{H}_G = \widehat{\bigoplus}_{s \in \mathrm{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s)$$

where $\mathcal{H}_G(s), s \in \mathrm{Spec}(\mathbb{H}_G)$, *are the finite-dimensional joint eigenspaces of* $\mathbb{H}_G$ *in* $\mathcal{H}_G$.

At the moment, we only have a conjectural description of $\mathrm{Spec}(\mathbb{H}_G)$ for $F = \mathbb{C}$ (and, in some cases, for $F = \mathbb{R}$).

So, let's go back to the case $F = \mathbb{C}$. Then we also have the algebra $\mathcal{A}_G = D_G \otimes \overline{D}_G$ of differential operators.

Observe that $\mathcal{A}_G$ acts on the space $V_G^\vee$ of distributions on Bun_G^{rs}, and $\mathcal{H}_G$ is naturally realized as a subspace of $V_G^\vee$. Hence we can apply elements of $\mathcal{A}_G$ to vectors in the eigenspaces $\mathbb{H}_G(s)$ of the Hecke operators, viewed as *distributions*.

Conjecture 8

Every $\mathbb{H}_G(s)$ is an eigenspace of $\mathcal{A}_G$.

Corollary 9

If $(\chi, \mu) \in \mathrm{Spec}\, \mathcal{A}_G$, *then* $\mu = \tau(\overline{\chi})$ *and* $\chi \in \mathrm{Op}^{\gamma}_{^LG}(X)_{\mathbb{R}}$.

Recall that $\mathrm{Op}_{^LG}(X)_{\mathbb{R}}$ is the subset of real ^LG-opers in $\mathrm{Op}_{^LG}(X)$.

Remark. Recall that first we defined a Hecke operator $\widehat{H}_\lambda(x) : V_G(\lambda) \to V_G$.

The algebra $\mathcal{A}_G$ naturally acts on both $V_G(\lambda)$ and V_G. Hence the commutators $[P, \widehat{H}_\lambda(x)], P \in \mathcal{A}_G$, make sense.

We have $[P, \widehat{H}_\lambda(x)] = 0, \quad \forall P \in \mathcal{A}_G$.

To see this, realize Bun_G as $G(X\backslash x)\backslash G(F_x)/G(\mathcal{O}_x)$.

Then $\widehat{H}_\lambda(x)$ acts from the *right*, whereas $\mathcal{A}_G$ can be obtained from the action of the center of $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{crit}}$ from the *left*.

However, to prove Conjecture 8 we need a stronger form of commutativity, and a crucial element in proving it is the system of differential equations satisfied by $\widehat{H}_\lambda(x)$.

The case of $G = PGL_2$, so ${}^LG = SL_2$

Consider SL_2-opers on X (following Beilinson and Drinfeld):

$$\mathrm{Op}_{SL_2}(X) = \bigsqcup_{\gamma \in \theta(X)} \mathrm{Op}^{\gamma}_{SL_2}(X)$$

where $\theta(X)$ is the set of isomorphism classes of square roots of K_X.

Pick a square root $K_X^{1/2}$ of K_X. An SL_2-oper in the corresponding component $\mathrm{Op}^{\gamma}_{SL_2}(X)$ is a holomorphic connection on the rank 2 vector bundle $\mathcal{V}_{\omega_1}$

$$0 \to K_X^{1/2} \to \mathcal{V}_{\omega_1} \longrightarrow K_X^{-1/2} \to 0$$

satisfying a *transversality condition*.

Here's an alternative description of this component.

A *projective connection* associated to $K_X^{1/2}$ is a second-order differential operator $P : K_X^{-1/2} \to K_X^{3/2}$ such that

1. $\mathrm{symb}(P) = 1 \in \mathcal{O}_X$, and
2. P is algebraically self-adjoint.

They form an affine space $\mathcal{P}roj_\gamma(X)$. Locally, $P = \partial_z^2 - v(z)$.

Lemma 10

There is a bijection $\mathrm{Op}_{SL_2}^\gamma(X) \simeq \mathcal{P}roj_\gamma(X)$

$$\chi \in \mathrm{Op}_{SL_2}^\gamma(X) \quad \mapsto \quad P_\chi \in \mathcal{P}roj_\gamma(X)$$

such that the section $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ *corresponding to the embedding* $K_X^{1/2} \hookrightarrow \mathcal{V}_{\omega_1}$ *satisfies* $P_\chi \cdot s_{\omega_1} = 0$
(here we use the $\mathcal{D}_X$*-module structure on* $\mathcal{V}_{\omega_1}$ *corresponding to* ∇_χ*).*

Let $\mathcal{V}_{\omega_1}^{\text{univ}}$ be the universal vector bundle over $\text{Op}_{SL_2}^{\gamma}(X) \times X$ with a partial connection ∇^{univ} along X, such that

$$(\mathcal{V}_{\omega_1}^{\text{univ}}, \nabla^{\text{univ}})|_{\chi \times X} = (\mathcal{V}_{\omega_1}, \nabla_{\chi})$$

Let $\mathcal{V}_{\omega_1,X}^{\text{univ}} := \pi_*(\mathcal{V}_{\omega_1}^{\text{univ}})$, where $\pi : \text{Op}_{SL_2}^{\gamma}(X) \times X \to X$. The connection ∇^{univ} makes $\mathcal{V}_{\omega_1,X}^{\text{univ}}$ into a left $\mathcal{D}_X$-module.

The algebra $D_{PGL_2} \simeq \text{Fun}\, \text{Op}_{SL_2}^{\gamma}(X)$ acts on $\mathcal{V}_{\omega_1,X}^{\text{univ}}$ and commutes with the action of $\mathcal{D}_X$.

Lemma 11

There is a unique second-order differential operator

$$\sigma : K_X^{-1/2} \to D_{PGL_n} \otimes K_X^{3/2}$$

satisfying the following property: for any $\chi \in \text{Op}_{SL_2}^{\gamma}(X)$, applying the corresponding homomorphism $D_{PGL_2} \to \mathbb{C}$ we obtain P_{χ}.

Differential equation on Hecke operators

As x varies along X, the Hecke operators $\widehat{H}_{\omega_1}(x)$ combine into a section of the C^∞ line bundle $|K_X|^{-1/2}$ on X with values in operators $\mathcal{H}_{PGL_2} \to \mathcal{H}_{PGL_2}$. We denote it by $\widehat{H}_{\omega_1}$.

Theorem 12

The Hecke operator $\widehat{H}_{\omega_1}$, viewed as an operator-valued section of $|K_X|^{-1/2}$, satisfies the system of differential equations

$$\sigma \cdot \widehat{H}_{\omega_1} = 0, \qquad \overline{\sigma} \cdot \widehat{H}_{\omega_1} = 0$$

This is a system of second-order differential equations (one holomorphic and one anti-holomorphic).

Explicitly, pick a point $\chi_0 \in \mathrm{Op}^{\gamma}_{SL_2}(X)$ and use it to identify $\mathrm{Op}^{\gamma}_{SL_2}(X)$ with $H^0(X, K_X^2)$.

Pick a basis $\{\varphi_i, i = 1, \ldots, 3g-3\}$ of $H^0(X, K_X^2)$

Let $\{F_i, i = 1, \ldots, 3g-3\}$ be the dual set of generators of the polynomial algebra $\mathrm{Fun}\,\mathrm{Op}^{\gamma}_{SL_2}(X) = D_{PGL_2}$ dual to this basis.

Each F_i is a global holomorphic diff. operator on Bun_{PGL_2}.

Locally on X, $P_{\chi_0} = \partial_z^2 - v_0(z)dz^2$. Then

$$\sigma = \partial_z^2 - v_0(z)dz^2 - \sum_{i=1}^{3g-3} F_i \otimes \varphi_i \ : \ K_X^{-1/2} \to D_{PGL_2} \otimes K_X^{3/2}$$