Relations Between Kazhdan-Lusztig Polynomials for Real and p-adic Groups

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Lie Groups Day
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(in honor of David Vogan)

Make no little plans...

OUTLINE (SOME SKETCHES)

Goal: Relate irreducible characters of representations of real and p-adic reductive groups, e.g. split classical groups.

1. What could one mean by this? Short answer: find relations among the "decomposition numbers" relating irreducible characters to characters of standard modules. These are given by the Kazhdan-Lusztig polynomials of the title, so one really has to relate the geometry that defines them.
2. Geometry of the LLC over $\mathbb{R}$.
3. Geometry of LLC over $\mathbb{Q}_p$.
4. Main results for certain classical split groups (joint with Leticia Barchini extending earlier joint work with Dan Ciubotaru): strong geometric relations between (2) and (3) that imply the KL polynomials that appear in the p-adic case are a computable subset of those for the real case.

IRREDUCIBLE CHARACTERS FOR $G_{\mathbb{R}}$ (SOME PLANS)

Fix $G_{\mathbb{R}}$ with maximal compact subgroup $K_{\mathbb{R}}$. Harish-Chandra determined the characters of the discrete series of $G_{\mathbb{R}}$.

Then work of many people showed how to explicitly compute the characters of parabolically induced representations,

$$\mathrm{Ind}_{M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}}^{G_{\mathbb{R}}}(\sigma \otimes \nu \otimes 1),$$

with σ a limit of discrete series. So characters of these representations are taken to be well-known.

The strategy is then to relate the irreducible characters to these standard characters.

LANGLANDS CLASSFICATION (SOME PARAMETERS)

Fix an infinitesimal character λ. There is a (finite) parameter set $\mathcal{P}ar_\lambda$ very roughly consisting of $G_{\mathbb{R}}$ conjugacy classes of pairs $\gamma = (M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}, \sigma \otimes \nu \otimes 1)$. To each γ one can associate

$$\mathrm{std}(\gamma) = \mathrm{Ind}_{M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}}^{G_{\mathbb{R}}}(\sigma \otimes \nu \otimes 1)$$

and a canonical irreducible subquotient $\mathrm{irr}(\gamma)$ inducing a bijection

$$\begin{aligned} \mathcal{P}ar_\lambda &\longrightarrow \widehat{G}_{\mathbb{R},\lambda} = \{\text{irreps of } G_{\mathbb{R}}, \text{ inftl. char } \lambda\} \\ \gamma &\mapsto \mathrm{irr}(\gamma) \end{aligned}$$

As characters (or in an appropriate K group), we write

$$[\mathrm{irr}(\gamma)] = \sum_{\delta \in \mathcal{P}ar_\lambda} m_{\gamma\delta}\, [\mathrm{std}(\delta)].$$

Computing irreducible characters amounts to computing the integers $m_{\gamma\delta}$. How?!

COMPUTING THE $m_{\gamma\delta}$ (SOME GEOMETRY)

For λ integral, one key idea is to interpret $\mathcal{P}ar_\lambda$ as (certain) irreducible $K = (K_\mathbb{R})_\mathbb{C}$ equivariant local systems on the complex flag variety $\mathcal{B}$.

To each $\gamma \in \mathrm{Loc}'_K(\mathcal{B})$, one may consider a constructible sheaf $\mathrm{con}(\gamma)$ and its perverse extension $\mathrm{per}(\gamma)$. Then, in an appropriate K group, write

$$[\mathrm{per}(\gamma)] = \sum_{\delta \in \mathrm{Loc}'_K(\mathcal{B})} m^g_{\gamma\delta}\,[\mathrm{con}(\delta)].$$

Vogan, Lusztig-V: These geometric multiplicities match the representation theoretic ones, and they are effectively computable,

$$\boxed{m_{\gamma\delta} = \pm m^g_{\gamma\delta}}\,.$$

Remember we are trying to match the coefficients $m_{\gamma\delta}$ with their p-adic counterparts. But this is hopeless because there is no p-adic analog of $\mathrm{Loc}_K(\mathcal{B})$. We need to see the dual group.

First try: interpret $\mathcal{P}ar_\lambda$ as (certain) irreducible $G^\vee$ equivariant local systems on the space of real Langlands parameters,

$$\left\{\phi \,:\, W_{\mathbb{R}} \to G^\vee \mid ...\right\} \,.$$

The problem is that every $G^\vee$ orbit is closed so $\mathrm{per}(\gamma) = \mathrm{con}(\gamma)$ for all γ. The geometry isn't rich enough.

Adams-Barbasch-Vogan modified the space of Langlands parameters to remedy this.

ADAMS-BARBASCH-VOGAN GEOMETRY (SOME FRIENDS)

ABV reinterpet $\mathcal{P}ar_\lambda$ as irreducible $G^\vee$ equivariant local systems on an ABV space $\mathcal{X}_\lambda$. To each $\gamma, \delta \in \mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda)$, one can write

$$[\mathrm{con}(\gamma)] = \sum_\delta M^g_{\gamma\delta}\,[\mathrm{per}(\delta)].$$

ABV: geometric multiplicities match the representation theoretic ones, and they are effectively computable,

$$\boxed{m_{\gamma\delta} = \pm M^g_{\delta\gamma}}\,.$$

Note this is fundamentally different from the Lusztig-Vogan setting.

ABV SPACE $\mathcal{X}_\lambda$ (SOME MORE PARAMETERS)

Assume $\lambda \in \mathfrak{h}^*$ integral, real; think of $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}^\vee$. Let

$$\mathfrak{p}^\vee(\lambda) \text{ be the sum of the nonnegative eigenspaces of } \mathrm{ad}(\lambda),$$

and $\mathcal{P}^\vee(\lambda)$ for the conjugates of $\mathfrak{p}^\vee(\lambda)$. Then, roughly, there is a symmetric subgroup $K^\vee$ of $G^\vee$ such that

$$\mathcal{X}_\lambda = G^\vee \times_{K^\vee} \mathcal{P}^\vee(\lambda) = G^\vee \times \mathcal{P}^\vee(\lambda) \big/ (gk, x) \sim (g, kx).$$

There is a canonical bijection $\mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda) = \mathrm{Loc}_{K^\vee}(\mathcal{P}^\vee(\lambda))$ that preserves all geometric information.

Upshot: The decomposition numbers for $G_\mathbb{R}$ are controlled by the $K^\vee$ orbits on $\mathcal{P}^\vee(\lambda)$.

THE TERMINOLOGY (SOME EXPLANATIONS)

LIE ALGEBRA COHOMOLOGY AND

THE REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

by

David Vogan

B.A. University of Chicago

(1974)

S.M. University of Chicago

(1974)

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3.

Table of Contents

Acknowledgements

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Finally, I would like to thank Professor Paul Sally of the University of Chicago, who undertook the formidable task of introducing me to representation theory while I was an undergraduate. It is not clear whether he will approve of the absence of analysis in this thesis, but nonetheless it is dedicated to him.

Assume $\lambda \in \mathfrak{h}^*$ integral; think of $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}^\vee$. Let

$$\mathfrak{p}^\vee(\lambda) \text{ be the sum of the nonnegative eigenspaces of } \mathrm{ad}(\lambda),$$

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$$\mathcal{X}_\lambda = G^\vee \times_{K^\vee} \mathcal{P}^\vee(\lambda) = G^\vee \times \mathcal{P}^\vee(\lambda)/(gk, x) \sim (g, kx).$$

There is a canonical bijection $\mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda) = \mathrm{Loc}_{K^\vee}(\mathcal{P}^\vee(\lambda))$ that preserves all geometric information.

Upshot: The decomposition numbers for $G_\mathbb{R}$ are controlled by the $K^\vee$ orbits on $\mathcal{P}^\vee(\lambda)$.

Suppose $H \subset G$ acts with finitely many orbits on Y. Consider

$$X = G \times_H Y = \{(g, y) \mid g \in G, y \in Y\} \big/ (gh, y) \sim (g, hy).$$

Then G acts via $g \cdot (g', y) = (gg', y)$, and the G orbits on X are bijection with the H orbits on Y,

$$H \cdot y \mapsto G \cdot (e, y),$$

and centralizers match: $G^{(e,y)} = H^y$. So have bijection $\gamma \mapsto \gamma'$

$$\mathrm{Loc}_G(X) = \mathrm{Loc}_H(Y).$$

Simple transversality considerations imply isomorphisms of intersection homology groups (and characteristic cycles),

$$M^g_{\gamma\delta} = M^g_{\gamma'\delta'}.$$

For a split group over $F = \mathbb{Q}_p$ with finite reside field $k = \mathbb{F}_p$, a Langlands parameter is roughly a representations

$$\varphi \,:\, \Gamma = \mathrm{Gal}(\overline{F}/F) \to G^\vee.$$

Choice of $\overline{F}$ gives a choice of $\overline{k}$, and so we get a surjective map

$$\Gamma \to \mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}}.$$

The kernel is called the inertia group I_F. Consider the preimage of $\mathbb{Z} = \langle x \mapsto x^p \rangle \subset \widehat{\mathbb{Z}}$; this defines the Weil group of F:

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1.$$

Weil-Deligne: incorporate the norm map on W_F ($\|w\| = p^n$ whenever w maps to $n \in \mathbb{Z}$) as follows,

$$W_F' = \mathbb{C} \ltimes W_F$$

where $wzw^{-1} = \|w\|$. A Langlands parameter is roughly a map

$$\varphi \,:\, W_F' \to G^\vee.$$

UNRAMIFIED p-ADIC PARAMETERS

Enhance the idea of a Galois representation

$$\varphi \ : \ \Gamma = \mathrm{Gal}(\overline{F}/F) \to G^\vee .$$

Weil group:

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1.$$

Weil-Deligne:

$$W'_F = \mathbb{C} \ltimes W_F \qquad\qquad wzw^{-1} = \| w \| \ .$$

An unramified Langlands parameter:

$$\varphi \ : \ W'_F \to G^\vee .$$

is one that is trivial on I_F. Determined by image of 1 (semisimple element) and Frob (unipotent element).

UNRAMIFIED p-ADIC PARAMETERS, CONTINUED

Rescaling and assuming λ real...following Lusztig, the space of unramified parameter for a split p-adic group consists of pairs

$$\left\{(\lambda, N) \mid \lambda \in \mathfrak{g}^\vee \text{ semisimple}, N \text{ nilpotent}\right\}$$

satisfying

$$[\lambda, N] = -N.$$

$G^\vee$ acts in the obvious way. If we fix λ, we are then talking about $L^\vee(\lambda) = (G^\vee)^\lambda$ orbits on $\mathfrak{g}^\vee_{-1}(\lambda)$.

Or, in other words, the space is

$$\mathcal{X}^F_\lambda = G^\vee \times_{L^\vee(\lambda)} \mathfrak{g}^\vee_{-1}(\lambda).$$

(Compare the ABV space $\mathcal{X}_\lambda = G^\vee \times_{K^\vee} \mathcal{P}^\vee(\lambda)$.)

Works perfectly: KL proof of the Deligne-Langlands conjecture and Lusztig's generalization to unipotent representations.

Let G_F be a split connected (adjoint) group over $F = \mathbb{Q}_p$, λ real. To (certain) $\gamma \in \mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda^F)$ there is a standard representations $\mathrm{std}(\gamma)$ with a canonical irreducible subquotient $\mathrm{irr}(\gamma)$ inducing a bijection

$$\mathrm{Loc}'_{G^\vee}(\mathcal{X}_\lambda^F) \longrightarrow \widehat{G}_{F,\lambda}^{unram}$$
$$\gamma \mapsto \mathrm{irr}(\gamma)$$

As characters (or in an appropriate K group), we write

$$[\mathrm{irr}(\gamma)] = \sum_{\delta \in \mathcal{P}ar_\lambda} m_{\gamma\delta}\,[\mathrm{std}(\delta)].$$

As before, computing irreducible characters amounts to computing the integers $m_{\gamma\delta}$.

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$$[\mathrm{irr}(\gamma)] = \sum_{\delta \in \mathcal{P}ar_\lambda} m_{\gamma\delta} \, [\mathrm{std}(\delta)].$$

As before, computing irreducible characters amounts to computing the integers $m_{\gamma\delta}$.

To each $\gamma, \delta \in \mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda^F)$, on can write

$$[\mathrm{con}(\gamma)] = \sum_\delta M_{\gamma\delta}^{g,F} \, [\mathrm{per}(\delta)].$$

Lusztig: geometric multiplicities match the representation theoretic ones, and they are *effectively* computable,

$$\boxed{m_{\gamma\delta}^F = \pm M_{\delta\gamma}^{g,F}} \, ,$$

just like the ABV case. But the effective algorithm is totally different to implement (Ciubotaru, ...)

Fix $\lambda \in \mathfrak{g}^\vee$ semisimple, real integral. Would like a natural injection

$$\mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda^F) \longrightarrow \mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda)$$
$$\gamma \mapsto \gamma'$$

that matches geometric multiplicities

$$[\mathrm{con}(\gamma)] = \sum_\delta M_{\gamma\delta}\, [\mathrm{per}(\delta)].$$

iff

$$[\mathrm{con}(\gamma')] = \sum_{\delta'} M_{\gamma\delta}\, [\mathrm{per}(\delta')] + \text{other unrelated terms},$$

and hence representation theoretic multiplicities.

Fix $\lambda \in \mathfrak{g}^\vee$ semisimple, integral. If we found a natural injection

$$\begin{gathered}\mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda^F) \longrightarrow \mathrm{Loc}_{G^\vee}(\mathcal{X}_\lambda)\\ \gamma \mapsto \gamma'\end{gathered}$$

that matches geometric multiplicities, then

$$[\mathrm{irr}(\gamma)] = \sum_\delta m_{\gamma\delta}\,[\mathrm{std}(\delta)],$$

an identity of virtual unipotent representations, iff

$$[\mathrm{irr}(\gamma')] = \sum_{\delta'} m_{\gamma\delta}\,[\mathrm{std}(\delta')] + \text{other unrelated terms},$$

an identify of virtual Harish-Chandra modules. Beware the exceptional groups....

THE REPRESENTATION THEORY UPSHOT II

Fix $\lambda \in \mathfrak{g}^\vee$ semisimple, integral. If we found a natural injection

$$\mathrm{Loc}_{G^\vee}(G^\vee \times_{L^\vee(\lambda)} \mathfrak{g}_{-1}(\lambda)) \longrightarrow \mathrm{Loc}_{G^\vee}(G^\vee \times_{K^\vee(\lambda)} \mathcal{P}(\lambda))$$
$$\gamma \mapsto \gamma'$$

that matches geometric multiplicities, then

$$[\mathrm{irr}(\gamma)] = \sum_{\delta} m_{\gamma\delta}\, [\mathrm{std}(\delta)],$$

an identity of virtual unipotent representations, iff

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THE REPRESENTATION THEORY UPSHOT III

Fix $\lambda \in \mathfrak{g}^\vee$ semisimple, integral. If we found a natural injection

$$\mathrm{Loc}_{L^\vee(\lambda)}(\mathfrak{g}_{-1}(\lambda)) \longrightarrow \mathrm{Loc}_{K^\vee(\lambda)}(\mathcal{P}(\lambda))$$
$$\gamma \mapsto \gamma'$$

that matches geometric multiplicities, then

$$[\mathrm{irr}(\gamma)] = \sum_{\delta} m_{\gamma\delta}\, [\mathrm{std}(\delta)],$$

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Theorem

Suppose $G = GL(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Fix an integral semisimple element $\lambda \in \mathfrak{g}$. Write $\mathfrak{g}_i$ for the i-eigenspace of $\mathrm{ad}(\lambda)$. Let L be the subgroup of G corresponding to $\mathfrak{g}_0$, and let $\mathcal{P}$ denote the variety of parabolics conjugate to $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. Set $\theta = Ad(\exp(i\pi\lambda))$ and $K = G^\theta$. Then there is an injection

$$\mathrm{Loc}_L(\mathfrak{g}_{-1}) \longrightarrow \mathrm{Loc}_K(\mathcal{P})$$
$$\gamma \mapsto \gamma'$$

such that

$$\textit{multiplicity of } \mathrm{con}(\gamma) \textit{ in } \mathrm{per}(\delta)$$

equals

$$\textit{multiplicity of } \mathrm{con}(\gamma') \textit{ in } \mathrm{per}(\delta').$$

(1) The $GL(n)$ case is joint with Ciubotaru from about 2010. (More on this in a moment.) In fact, earlier Lusztig and Zelvinsky had shown how to embed

$$\mathrm{Loc}_L(\mathfrak{g}_{-1}) \hookrightarrow S_n$$

and match decomposition numbers for highest weight modules (where there is no possibility of incorporating nontrivial local systems). The new result is for other classical groups which can capture those nontrivial local systems.

(2) Similar statements appear to be available for $\mathrm{SO}(n)$, but K depends not only on λ but also on γ. This is needed to account for the different blocks.

(3) Nothing so simple is possible for the exceptional groups.

Start with the $GL(n)$ case, λ integral. Then

$$\mathfrak{k} = \sum_{i\ even} \mathfrak{g}_i(\lambda) \simeq \mathfrak{gl}(\lfloor n/2 \rfloor) \oplus \mathfrak{gl}(\lceil n/2 \rceil)$$

contains $\mathfrak{l} = \mathfrak{g}_0(\lambda)$. No nontrivial local systems, so looking for a map

$$\{L \text{ orbits on } \mathfrak{g}_{-1}\} \longrightarrow \{K \text{ orbits on } \mathcal{P}_\lambda = G/P(\lambda)\}$$

Simplest possible idea: find an L-equivariant map,

$$\Phi \ : \ \mathfrak{g}_{-1} \longrightarrow \mathcal{P}_\lambda \simeq G/P(\lambda)$$

and define

$$L \cdot N \mapsto K \cdot \Phi(N).$$

The simplest possible Φ works,

$$\Phi(N) = (1 + N) \cdot \mathfrak{p}(\lambda).$$

Note that $\exp(N)$ doesn't work! So what to do for other (classical) groups?

SO WHAT'S THE MAP IN GENERAL?

Nothing like $\Phi(N) = (1+N)\cdot\mathfrak{p}$ exists in general (and can't). Want to explain theorem (with Barchini) for

$$G' = Sp(2n) \subset G = GL(2n), \quad \lambda \in \mathfrak{g}' \subset \mathfrak{g}.$$

1. First step: revisit CT and recognize
$$K \cdot \Phi(\mathfrak{g}_{-1}) \cdot \mathfrak{p} \simeq K \times_L \mathfrak{g}_{-1}$$
in a suitable affine open.
2. Second step: Take subbundle $K \times_L \mathfrak{g}'_{-1}$ and intersect with a cleverly chosen copy of $G'/P(\lambda)'$ in $G/P(\lambda)$.
3. Third step: recognize the intersection as $K' \times_{L'} \mathfrak{g}'_{-1}$, on one hand, and (dense in) matching K' orbits on $\mathcal{P}'$ on the other. That gives the result. Hardest part is showing the intersection is nonempty, and this requires a different analysis for $SO(n)$.

EXAMPLE OF $\lambda = \rho^\vee \ldots$

Set $T =$ diagonal in $G = \mathrm{GL}(n, \mathbb{C})$, $\lambda = \rho^\vee \in \mathfrak{t}$,

$$\rho^\vee = ((n-1)/2, (n-3)/2, \ldots, -(n-3)/2, -(n-1)/2)$$

$L(\lambda) = T$, $K = \mathrm{GL}(\lceil \frac{n}{2} \rceil) \times \mathrm{GL}(\lfloor \frac{n}{2} \rfloor)$.

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & & & \\ \star & 0 & & \\ & \star & \ddots & \\ & & \star & 0 \end{pmatrix} \qquad \mathfrak{p} = \begin{pmatrix} \star & \star & \star & \star \\ & \star & \star & \star \\ & & \ddots & \star \\ & & & \star \end{pmatrix}$$

So $T = L(\lambda)$ orbits on $\mathfrak{g}_1(\rho^\vee)$ are parameterized by subsets S of the simple roots, and all orbit closures are smooth. The closure of

$$K \cdot (1 + N_S) \cdot \mathfrak{p}$$

is smooth along all other orbits of the form $K \cdot (1 + N_{S'})$. This $\lambda = \rho^\vee$ case is generalized to all types with Barchini.