David Vogan and the unitary dual in 2022

The beachhead of Arthur's unipotent representations

Stephen D. Miller
Rutgers University

MIT Lie Groups Day
September 23, 2022

The last major unsolved problem…

- …in the representation theory of real reductive Lie groups G…
- is that of describing $\hat{G}$ = {unitary representations of G}
  - Representations (π, V) which preserve a Hilbert space inner product
- E.g., if G is compact then:
  - all representations are known (Weyl's "theorem of the highest weight")
  - all are unitary (Weyl's "Unitarian trick", cf. Hurwitz)
- Must decide if a given representation is unitarizable
- Gets harder as group gets less compact
- *Split groups* like $SL(n,\mathbb{R}), E_8(\mathbb{R})$ are hardest. E.g.,
  - Bargmann (1947): $\widehat{SL(2,\mathbb{R})}$
  - Vogan (1986): $\widehat{SL(n,\mathbb{R})}, \widehat{SL(n,\mathbb{C})}$
  - Vogan (1994): $\widehat{G_2(\mathbb{R})}$
- These are major accomplishments of David Vogan from the period *before* he retired

The really big goal

- A full understanding of $\widehat{G}$ for any real reductive Lie group G
- $\widehat{E_8(\mathbb{R})}$ would (perhaps?) be nearly as satisfactory.
- Special cases (e.g., "spherical" – has a vector fixed under action of maximal compact subgroup) important for number theory.
- Question posed to Vogan at Rutgers' Faculty Dining Hall (Nov. 2015):
  - Are we ever going to see this?
- Vogan's answer:
  - I'd like to see it at least settled for a large, natural class of representations
  - For example, those coming up in automorphic forms
  - Specifically, Arthur's unipotent representations
    - Very vague folk mythology: are building blocks of the unitary dual

What are Arthur's unipotent representations?

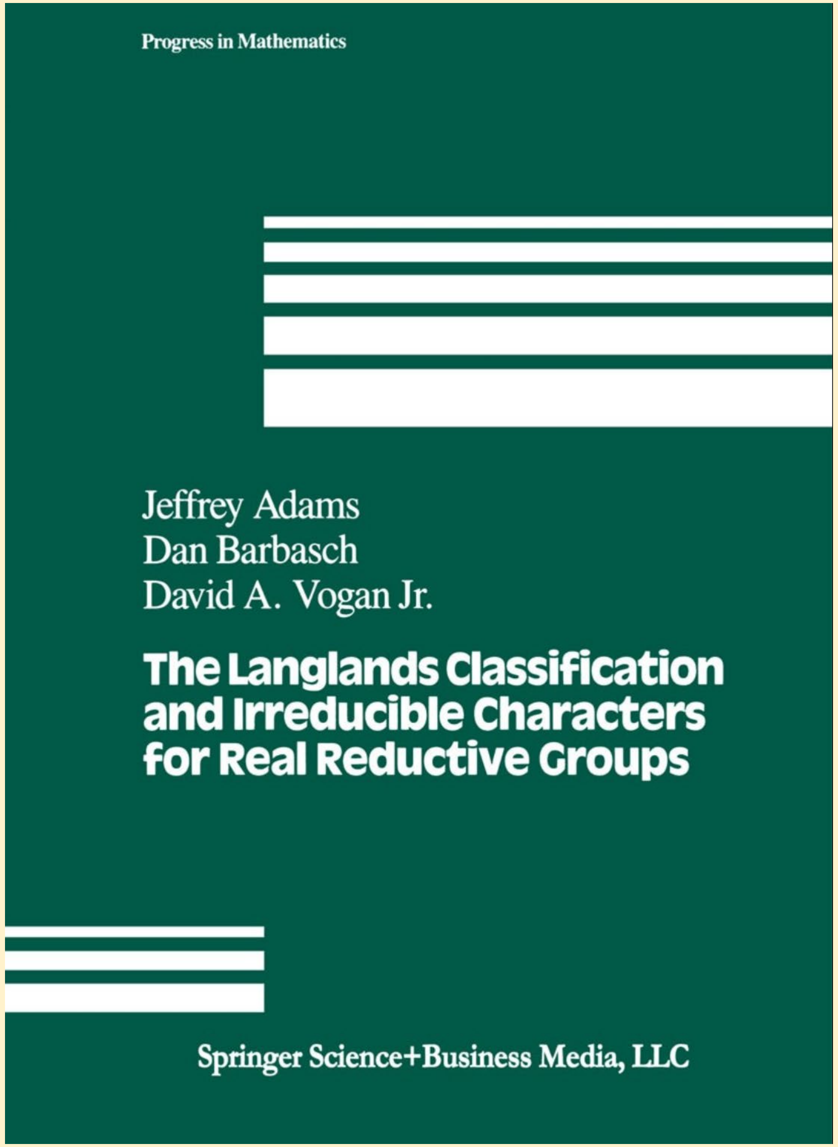
- Hard question – Arthur describes indirectly
- [ABV] book gives an answer compatible w/ Arthur
- We'd like to identify them
- Should be related to automorphic forms, but very unclear how

- Two parts of the definition [expanded on next slides]:
  - Notion of Arthur parameter
  - Notion of Arthur packets

- [ABV] proves all major expected properties of these packets aside from one:
  - **UNITARITY!**

- Celebrated application of The Fundamental Lemma: Arthur proves unitarity for classical groups SL(n), SO(n,n), SO(n+1,n), Sp(2n),….
  - but not Spin covers

- **Main theorem today** [Adams-van Leeuwen-M-Vogan]: **unitarity proven for *all* real forms of exceptional groups, including $E_8(\mathbb{R})$.**



Arthur parameters

- Let G = split real group, e.g., $SL(2,\mathbb{R}), SO(n,n), E_8(\mathbb{R})$, ...
- Let $G^\vee(\mathbb{C})$ = complex points of Langlands dual group, e.g., $PGL(2,\mathbb{C})$.
- *Weil group* $W_\mathbb{R} = \mathbb{C}^* \sqcup j\mathbb{C}^* \subseteq$ Quaternions*
- *Arthur parameter*: algebraic homomorphism $\psi: SL(2,\mathbb{C}) \times W_\mathbb{R} \to G^\vee(\mathbb{C})$
  - + technical hypothesis (irrelevant hereafter)
- *Unipotent* Arthur parameter: ψ is trivial on the $\mathbb{C}^*$-factor of $W_\mathbb{R}$.
  - So $\psi(j)^2 = \psi(-1) = 1$, i.e., $\psi(j)$ has order 1 or 2.
  - Up to conjugacy, determined by $\psi(j)$ and an algebraic $SL(2,\mathbb{C})$ which commutes with it in $G^\vee(\mathbb{C})$

- Meaning of the $SL(2,\mathbb{C})$ for a representation associated to ψ:
  - "how far away from tempered" (not always true anyhow) or
  - "wavefront set", measures how complicated representations are.
    - Also a measure of size of an infinite-dimensional representation [more later]

Nilpotent orbits (≈Jordan canonical form)

- Interesting to think of $\psi(SL(2,\mathbb{C}))$ as encoding a *nilpotent orbit* $\mathcal{O}^\vee$ of $G^\vee(\mathbb{C})$ under its adjoint action on its Lie algebra $\mathfrak{g}^\vee(\mathbb{C})$:
  - consider differential $d\psi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O}^\vee$
  - Part of an $\mathfrak{sl}_2$-triple, with neutral element $2\lambda \coloneqq d\psi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}^\vee \cong \mathfrak{a}^*$.
  - The adjoint nilpotent orbit $\mathcal{O}^\vee$ has weighted Dynkin diagram determined by 2λ.
- Up to conjugacy, unipotent parameters for *split* G determined by
  - order 1 or 2 element $\psi(j)$, and an
  - adjoint nilpotent orbit intersecting
    {+1 eigenspace of $Ad(\psi(j))$} = a symmetric subalgebra of $\mathfrak{g}^\vee(\mathbb{C})$
- More precise information determined by the nilpotent orbit $\mathcal{O}^\vee$:
  - its Spaltenstein dual orbit $\mathcal{O} \subset \mathfrak{g}^\vee(\mathbb{C})$ is ≈ wavefront set of representations associated to ψ
  - This is a measure of size of an infinite-dimensional representation
  - Small wavefront set $\implies$ delicate number-theory in automorphic form coefficients.

From unipotent Arthur parameters to packets

- To each parameter, Arthur predicts a *packet* of interesting **unitary** representations.
- How to define this packet?
  - Arthur: indirectly. Packets must involve stable combinations of characters
  - [Adams-Barbasch-Vogan, 1991]: give *a* definition of packets (for real groups).
    - Stability satisfied, but stability isn't enough to define a packet.
- Only one easy-to-define member of a packet: the "Langlands element"
  - quotient of (dominant) principal series with
    infinitesimal character $\lambda = \frac{1}{2}$ (weighted Dynkin diagram for $\mathcal{O}^\vee) \in \mathfrak{a}^*$
    and quadratic character defined by $\psi(j)$
- Unlike L-packets, Arthur packets can intersect nontrivially!
- Notion of weak packet (just as useful for unitarity): union of all packets with same λ (i.e., same $\mathcal{O}^\vee$)
- `atlas` software: method to compute weak packets using cell computations, character tables of (large!) Weyl groups.

Results for split exceptional groups

- Arthur: settled for many split classical groups using trace formula & fundamental lemma
  - $C(\mathcal{O})$ = component group of centralizer of $\mathcal{O}^\vee$ in $G^\vee(\mathbb{C})$ is always abelian
- List of all unipotent Arthur parameters for exceptional groups now known by [Hundley-M]
  - 341 total
  - $C(\mathcal{O})$ not always abelian (e.g., S_5), so intricate new phenomena enter
- All unipotent representations for exceptional groups recently computed [Adams-Van Leeuwen-M-Vogan]
- Previously studied packets (e.g., minimal or next-to-minimal representation) tend to be singletons = {Langlands element}, but many are big
- [M- 2012], [Hundley-M 2019] showed the Langlands element of the packet is always unitary, using Eisenstein series [more later] and, in some cases, the `atlas` software.
  - Arthur: all packet elements should have automorphic realizations, but it is not clear if as cusp forms or Eisenstein series residues.

Census of unipotent representations

G	K	#Unip
G_2(cpt)	$G2$	1
G_2(split)	$2A1$	12
F_4(compact)	$F4$	1
F_4 (B_4)	$B4$	3
F_4 (split)	$A1 + C3$	75
E_6^{sc} (compact)	$E6$	1
E_6^{sc}(herm.)	$D5 + T1$	12
E_6^{sc}(quasisplit)	$A1 + A5$	47
E_6^{sc}(F_4)	$F4$	3
E_6^{sc}(split)	$C4$	68
E_6^{ad}(compact)	E_6	1
E_6^{ad}(herm.)	$D5 + T1$	12
E_6^{ad}(quasisplit)	$A1 + A5$	47
E_6^{ad}(F_4)	$F4$	3
E_6^{ad}(split)	$C4$	68

G	K	#Unip
E_7^{sc}(compact)	E_7	1
E_7^{sc} (herm.)	$E_6 + T1$	28
E_7^{sc}(quat.)	$A1 + D6$	56
E_7^{sc}(split)	$A7$	252
E_7^{ad}(compact)	E_7	1
E_7^{ad}(herm.)	$E_6 + T1$	23
E_7^{ad}(quat.)	$A1 + D6$	54
E_7^{ad}(split)	$A7$	276
E_8(compact)	E_8	1
E_8(quat.)	$A1 + E_7$	57
E_8 (split)	$D8$	362
TOTAL		1,465

Three ways to show unitarity

1. `atlas` software: algorithm of Adams-van Leeuwen-Trapa-Vogan
    - Determines unitarity, but can require huge amounts of time and memory
    - Works very well for *most* of the examples on the census
    - Probably cannot help for very large orbits $\mathcal{O}^\vee$
    - Example: trivial representation on E_7 took several days and hundreds of GB of RAM. Only became possible in 2021.

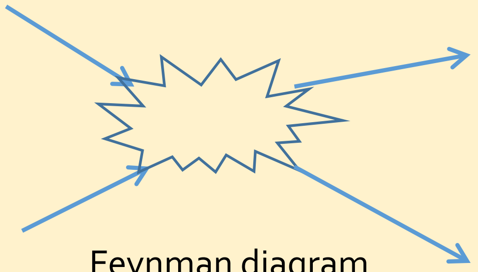
2. "Golf" – an analysis of induction and reducibility after deformation [Vogan]
    - Relies on the fact that the E_8 root lattice is absurdly dense
        - Best sphere packing [Viazovska], kissing number [Odlyzko-Sloane]
        - "Universally Optimal" under many measures [Cohn-Kumar-M-Radchenko-Viazovska]
    - so each representation has lots of neighbors that might be unitary
    - Unitarity might extend if no reducibility occurs in between.

3. Eisenstein series/string theory handles atlas' hardest cases

Who ordered string theory?

- Background:

<u>Green + Vanhove +… :</u> studied 4-graviton scattering amplitude in $N = 8$ (maximally symmetric) type IIB string theory.



Feynman diagram

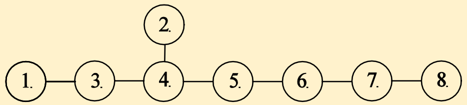
What is the probability distribution of the outgoing particles? How does it depend on the coupling constants? Automorphically!

- Basic idea [outline]:
  - String theory posits low-energy corrections to general relativity
  - [Green-M-Vanhove-Russo]: the first terms are Eisenstein series.
  - Like all automorphic forms, they generate representations
  - [Green-M-Vanhove]: these are the hardest unipotent representations for `atlas`.
  - String theory gives serious growth constraints
    - These constraints imply the Eisenstein series $\in L^2(\Gamma\backslash G)$, hence are unitarity!
  - This approach can be generalized and made rigorous.

4-graviton scattering amplitude

- Has contributions from “analytic” and “non-analytic” parts:

$$A_D(s,t,u) \;=\; A_D^{analytic}(s,t,u) \;+\; A_D^{nonanalytic}(s,t,u)$$



- The analytic part has an expansion in terms of momenta:

$$\begin{aligned} A^{analytic} &= \sum_{p=0,\,q=-1}^{\infty} \ell_D^6\, \mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})\, \sigma_2^p\, \sigma_3^q \\ &= \frac{3}{\sigma_3}\mathcal{R}^4 + \ell_D^6\, \mathcal{E}^{(D)}_{(0,0)}(\phi_{d+1})\,\mathcal{R}^4 + \ell_D^{10}\, \mathcal{E}^{(D)}_{(1,0)}(\phi_{d+1})\,\sigma_2\,\mathcal{R}^4 \\ &\quad + \ell_D^{12}\, \mathcal{E}^{(D)}_{(0,1)}(\phi_{d+1})\,\sigma_3\,\mathcal{R}^4 + \cdots , \end{aligned}$$

Einstein-Hilbert term

Low energy correction terms

Closed string theory in flat Minkowsky space-time of dimensions $3 \leqslant D \leqslant 10$ times a $d = 10 - D$ torus $\mathbb{T}^d$

D	$E_{d+1}(\mathbb{R})$	K_{d+1}	$E_{d+1}(\mathbb{Z})$
10A	$Gl(1,\mathbb{R})$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R}) \times \mathbb{R}^\times$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times SO(2)$	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$SO(16)$	$E_{8(8)}(\mathbb{Z})$

- E_{d+1} real split forms, K_{d+1} maximal compact subgroup.

- Supergravity: ϕ_{d+1} naturally lives inside a symmetric space E_{d+1}/K_{d+1}
- String theory: ϕ_{d+1} is also invariant under a discrete subgroup $\Gamma \subset E_{d+1}$
- Challenge: identify the automorphic coefficients $\mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})$

Identification of $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$

- Using the differential equations and relations between physics in dimensions D and D+1 (pertubative limit, M-theory limit, and decompatification limit), some asymptotics can be understood.
- [Green-M-Vanhove-Russo] found chain of solutions on different duality groups in terms of Eisenstein series.

The result:

$G_d(\mathbb{Z}) = E_{d+1}(\mathbb{Z})$	$\mathcal{E}^{(D)}_{(0,0)}$	$\mathcal{E}^{(D)}_{(1,0)}$
$E_{8(8)}(\mathbb{Z})$	$\mathbf{E}^{E_8}_{[1\,0^7];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_8}_{[1\,0^7];\frac{5}{2}}$
$E_{7(7)}(\mathbb{Z})$	$\mathbf{E}^{E_7}_{[1\,0^6];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_7}_{[1\,0^6];\frac{5}{2}}$
$E_{6(6)}(\mathbb{Z})$	$\mathbf{E}^{E_6}_{[1\,0^5];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_6}_{[1\,0^5];\frac{5}{2}}$
$SO(5,5,\mathbb{Z})$	$\mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}}$	$\frac{1}{2}\,\hat{\mathbf{E}}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{4}{45}\hat{\mathbf{E}}^{SO(5,5)}_{[00001];3}$
$SL(5,\mathbb{Z})$	$\mathbf{E}^{SL(5)}_{[1000];\frac{3}{2}}$	$\frac{1}{2}\,\hat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{\pi}{30}\hat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}}$
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$\hat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\hat{\mathbf{E}}_1(U)$	$\frac{1}{2}\,\mathbf{E}^{SL(3)}_{[10];\frac{5}{2}} - 4\,\mathbf{E}^{SL(3)}_{[10];-\frac{1}{2}}\,\mathbf{E}_2(U)$
$SL(2,\mathbb{Z})$	$\mathbf{E}_{\frac{3}{2}}(\Omega)\,\nu_1^{-\frac{3}{7}} + 4\zeta(2)\,\nu_1^{\frac{4}{7}}$	$\frac{1}{2}\,\nu_1^{-\frac{5}{7}}\,\mathbf{E}_{\frac{5}{2}}(\Omega) + \frac{2\zeta(2)}{15}\,\nu_1^{\frac{9}{7}}\mathbf{E}_{\frac{3}{2}}(\Omega) + \frac{4\zeta(2)\zeta(3)}{15}\,\nu_1^{-\frac{12}{7}}$
$SL(2,\mathbb{Z})$	$\mathbf{E}_{\frac{3}{2}}(\Omega)$	$\frac{1}{2}\,\mathbf{E}_{\frac{5}{2}}(\Omega)$

← Simpler

(the constant function is the value at s=0 of any of these series, so it too is an Eisenstein series).

What are Eisenstein series? First, $SL(2)$

- Eisenstein introduced the holomorphic modular forms of Im(z)>0

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-k}$$

- Siegel defined the non-holomorphic analog for $z = x + iy, y > 0$

$$E(x+iy,s) \coloneqq \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}}$$

- The latter has a Fourier expansion

$$E(x+iy,s) = y^s + c(2s-1)y^{1-s} + \sum_{n\neq 0} e^{2\pi inx}\, C_n(y,s)$$

- where $c(s), C_n(y,s)$ given by explicit number-theoretic formulas
- Growth controlled by constant term $y^s + c(2s-1)y^{1-s}$
- Application: Residue at $s = 1$ is *constant*
  - This is square-integrable over modular fundamental domain vs. $y^{-2}dx\, dy$

For more general groups

- First consider a spherical principal series

$$V_\lambda = \{f : G \to \mathbb{C} \mid f(nag) = e^{(\lambda+\rho)(\log(a))} f(g)\},$$

- For $\lambda - \rho$ strictly dominant ("Godement range") define

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{(\lambda+\rho)(H(\gamma g))}$$

- Satisfies functional equations $E(\lambda, g) = M(w, \lambda) E(w\lambda, g)$ where

$$M(w,\lambda) = \prod_{\substack{\alpha \in \Delta^+ \\ w\alpha \in \Delta^-}} c(\langle \lambda, \alpha^\vee \rangle) \qquad c(s) := \frac{\xi(s)}{\xi(s+1)} \quad \text{and} \quad \xi(s) := \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s)$$

and $w \in W$ = Weyl group.

- Langlands' constant term formula

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\, dn = \sum_{w \in W} M(w, \lambda)\, e^{(w\lambda+\rho)(H(g))}$$

Residues at special points

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)\, dn \;=\; \sum_{w\in W} M(w,\lambda)\, e^{(w\lambda+\rho)(H(g))}$$

$$M(w,\lambda) \;=\; \prod_{\substack{\alpha\in\Delta^+ \\ w\alpha\in\Delta^-}} c(\langle\lambda,\alpha^\vee\rangle)$$

- Previous $SL(2)$ example was at $\lambda = \rho$
- In general $E(-\rho, g) \equiv 1$ (constant function for any G)
- Residual spectrum gets *very* interesting and intricate for other groups
- Get L^2 residue if nontrivial contributions satisfy $\boldsymbol{\langle w\lambda, \alpha_i^\vee\rangle < 0, \forall i}$ (Langlands' condition).
- [M-,2012]: specialize deformation $\lambda = 2s\varpi_j - \rho$ to compute constant terms.
  - Corresponds to maximal parabolic Eisenstein series induced from trivial representation.
  - Actually, all of Arthur's spherical examples have this form.
  - Later [M-Hundley] showed the "basepoint" representation in each Arthur packet is unitary
    - hardest one from the point of view of atlas
- **<u>Key point:</u>** The inner products $\langle\lambda, \alpha_i^\vee\rangle = -1$ for $i \neq j$. This rules out the contributions for nearly all $w \in W$. (Most $M(w,\lambda) = 0$ since $c(-1) = 0$.)
- Consequence: unitarity (since L^2 defines Hilbert space structure) of all spherical Arthur packet members

Unitarity [Adams-van Leeuwen-M-Vogan]

- Compute all unipotent representations for exceptional groups (1,465)
- Use `atlas` algorithm [Adams-van Leeuwen-Trapa-Vogan] to show 1,435 are accessible (with big machines), verified to be unitary.
  - Of the remaining 30, 22 are Langlands elements
  - Known to be unitary by [Hundley-M]
- Vogan's "Golf" on the back 8:
  - uses parabolic and cohomological induction from unipotents on smaller groups
  - gets nearby unitary representations
    - choice of what to induce is influenced by string theory construction
  - these nearby induced, unitary representations are deformed to the one we want
  - deformation *potentially* crosses a root-wall of reducibility
  - `atlas` computation: it stays irreducible, though
    - Therefore unitarity doesn't change along the deformation.
- Conclusion: all unipotent representations of exceptional groups are unitary