Some Comments on the Structure of the Unitary Dual

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Problem of Unitary Dual

Let G be a complex connected reductive algebraic group. Write

$$\Pi_{u,sph}(G) = \{\text{irred unitary spherical } G\text{-representations}\}.$$

Problem of the Unitary Dual (complex spherical case)

Parameterize the set $\Pi_{u,sph}(G)$.

Some history:

- $\mathrm{GL}(2)$ (Gelfand-Naimark, 1947)
- $\mathrm{SL}(3)$, $\mathrm{Sp}(4)$, G_2 (Duflo, 1979)
- $\mathrm{GL}(n)$ (Vogan, 1986)
- $\mathrm{Sp}(2n)$, $\mathrm{SO}(n)$ (Barbasch, 1989)

Goal: give a conjectural description of $\Pi_{u,sph}(G)$ for all G.

Harish-Chandra bimodules

- A *Harish-Chandra bimodule* is a $U(\mathfrak{g})$-bimodule V such that the adjoint action of $\mathfrak{g}$

$$\mathfrak{g} \times V \to V, \qquad (\xi, v) \mapsto \xi v - v\xi$$

  integrates to a rational (i.e. locally finite) G-action.
- A HC bimodule is *spherical* if it contains a nonzero fixed vector for the adjoint G-action.
- Write

$$\mathrm{HC}(G) = \{\text{irred HC bimodules}\}$$
$$\mathrm{HC}_{sph}(G) = \{\text{irred spherical HC bimodules}\}$$

Unitary Harish-Chandra Bimodules

- Fix a *compact* real form $\sigma : \mathfrak{g} \to \mathfrak{g}$. Induces a conjugate-linear algebra involution $\sigma : U(\mathfrak{g}) \to U(\mathfrak{g})$.
- A Hermitian form $\langle \; , \; \rangle$ on a HC bimod V is *invariant* if

$$\langle xvy, w \rangle = \langle v, \sigma(y) w \sigma(x) \rangle, \qquad x, y \in U(\mathfrak{g}), \; v, w \in V.$$

- V is *Hermitian* if it admits a non-degenerate invariant Hermitian form.
- V is *unitary* if it admits a positive-definite invariant Hermitian form.
- Write

$$\mathrm{HC}_u(G) = \{\text{irred unitary HC bimodules}\}$$
$$\mathrm{HC}_{u,sph}(G) = \{\text{irred spherical unitary HC bimodules}\}$$

Harish-Chandra Bimodules

Theorem (Harish-Chandra, Duflo,...)

$$\begin{array}{ccccc}
\Pi_{u,sph}(G) & \hookrightarrow & \Pi_{sph}(G) & \hookrightarrow & \Pi(G) \\
\updownarrow & & \updownarrow & & \updownarrow \\
HC_{u,sph}(G) & \hookrightarrow & HC_{sph}(G) & \hookrightarrow & HC(G) \\
 & & \updownarrow & & \\
 & & \mathfrak{h}^*/W & &
\end{array}$$

Problem of Unitary Dual (Take 2)

Thus, we can regard $\Pi_{u,sph}(G)$ as a W-invariant subset of $\mathfrak{h}^*$.

Problem of Unitary Dual (algebraic formulation, complex spherical case)

Compute the W-invariant subset $\Pi_{u,sph}(G) \subset \mathfrak{h}^*$.

Remark

It is useful and customary to restrict to the case of 'real infinitesimal character', i.e. $X^*(H) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{h}^*$. One can easily reduce to this case via unitary induction.

What does $\Pi_{u,sph}(G)$ look like?

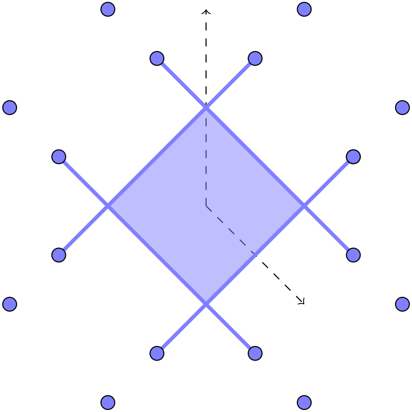
Some general features of $\Pi_{u,sph}(G)$:

- It is a *closed* subset of $\mathfrak{h}^*$ (in the Euclidean topology).
- It is contained in the closed ball $B(0, |\rho|)$ (probably a tighter bound is possible).
- It is a union of facets defined by certain hyperplanes in $\mathfrak{h}^*$ (roughly: affine co-root hyperplanes).

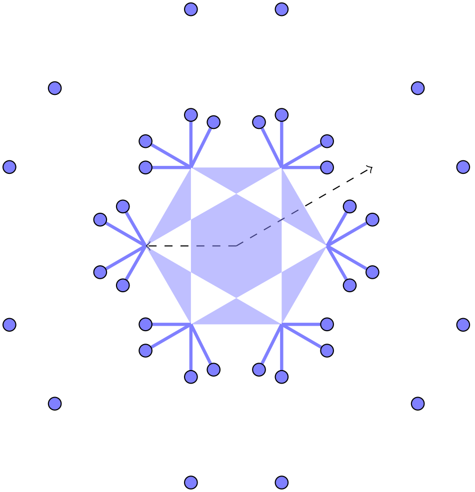
Ok, but what does it look like?

$\mathrm{SL}(2, \mathbb{C})$

$Sp(4, \mathbb{C})$



$G_2(\mathbb{C})$



How should we understand these pictures?

(1) Each picture contains a finite set of distinguished points: For each G, there is a finite set of reps (e.g. trivial, oscillator rep) called *unipotent representations*, which are unitary for magical reasons.

(2) Each picture contains copies of the pictures for its Levis: If $L \subset G$ is a Levi subgroup and $X_L \in \Pi_{u,sph}(L)$, then $\mathrm{Ind}_P^G X_L$ is unitary (and hence also its spherical summand).

(3) Each picture is closed under certain 'deformations': If $X \in \Pi_{u,sph}(G)$ belongs to a 'complementary series' C, then $C \subset \Pi_{u,sph}(G)$.

Vogan's Philosophy on the Unitary Dual ('Orange Book', 1987)

Every representation in $\Pi_{u,sph}(G)$ can be obtained by applying operations (2) and (3) to a unipotent representation (1) of a Levi subgroup $L \subset G$.

Vogan's Philosophy

In order to turn Vogan's philosophy into a precise mathematical conjecture, we need:

- a precise (and suitably general) definition of 'unipotent', and
- a precise (and suitably general) definition of 'complementary series'.

Claim: both goals are most naturally accomplished using the language of *filtered quantizations of nilpotent covers*.

Nilpotent covers

- A *nilpotent cover* for G is a finite, connected, G-equivariant cover of a nilpotent co-adjoint G-orbit. Write

$$\text{Cov}(G) = \{\text{nilpotent covers for } G\}/\sim$$

- If $\mathbb{O}$ is a nilpotent orbit and $e \in \mathbb{O}$, then covers of $\mathbb{O}$ are parameterized by conjugacy classes of subgroups of $A(\mathbb{O}) = Z_G(e)/Z_G(e)^\circ$.

Example: $\mathrm{SL}(2,\mathbb{C})$

Two nilpotent orbits: $\{0\}$ and $\mathbb{O} = G \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

- $A(\{0\}) = 1$. No nontrivial covers.
- $A(\mathbb{O}) = \mathbb{Z}_2$. One nontrivial (two-fold) cover.

Birational induction of nilpotent covers

- For each Levi subgroup $L \subset G$, there is a map
$$\mathrm{Bind}_L^G : \mathrm{Cov}(L) \to \mathrm{Cov}(G)$$
called *birational induction*.
- A cover is said to be *birationally rigid* if it cannot be obtained via birational induction from a proper Levi subgroup.
- A *birational induction datum* is a pair $(L, \widetilde{\mathbb{O}}_L)$ consisting of a Levi subgroup $L \subset G$ and a birationally rigid nilpotent cover $\widetilde{\mathbb{O}}_L$. Write
$$\Psi(G) = \{\text{birational induction data } (L, \widetilde{\mathbb{O}}_L)\}$$

Proposition (Losev, Matvieievskyi)

$$\mathrm{Bind} : \Psi(G)/G \xrightarrow{\sim} \mathrm{Cov}(G).$$

Quantizations of nilpotent covers

The ring of regular functions $\mathbb{C}[\widetilde{\mathbb{O}}]$ is a graded Poisson algebra. Can define *filtered quantizations* of $\mathbb{C}[\widetilde{\mathbb{O}}]$. Write

$$Q(\widetilde{\mathbb{O}}) := \{\text{filtered quantizations of } \mathbb{C}[\widetilde{\mathbb{O}}]\}/\sim$$

Choose $(L, \widetilde{\mathbb{O}}_L) \in \Psi(G)$ corresponding to $\widetilde{\mathbb{O}}$, and define

$$\mathfrak{h}(\widetilde{\mathbb{O}}) := \mathfrak{z}(\mathfrak{l} \cap [\mathfrak{g}, \mathfrak{g}])^*$$

Theorem (Losev, Losev-MB-Matvieievskyi)

There is a (finite) subgroup $W(\widetilde{\mathbb{O}}) \subset N_G(L)/L$ and a canonical bijection

$$\mathfrak{h}(\widetilde{\mathbb{O}})/W(\widetilde{\mathbb{O}}) \xrightarrow{\sim} Q(\widetilde{\mathbb{O}}), \qquad \lambda \mapsto \mathcal{A}_\lambda(\widetilde{\mathbb{O}})$$

Example

- Let $P = LU \subset G$ be a parabolic subgroup.
- There is a unique open G-orbit

$$\widetilde{\mathbb{O}} \subset T^*(G/P).$$

- Image of the moment map $T^*(G/P) \to \mathcal{N}$ is the closure of a nilpotent orbit (Richardson orbit for P).
- $\widetilde{\mathbb{O}}$ is a nilpotent cover, birationally induced from $(L, \{0\})$.
- $\mathfrak{h}(\widetilde{\mathbb{O}}) = \mathfrak{z}(\mathfrak{l} \cap [\mathfrak{g}, \mathfrak{g}])^*$.
- Given $\lambda \in \mathfrak{h}(\widetilde{\mathbb{O}})$, get TDO $\mathcal{D}_{G/P}^{\lambda+\rho(\mathfrak{u})}$ on G/P. Then

$$\mathcal{A}_\lambda(\widetilde{\mathbb{O}}) = \Gamma(G/P, \mathcal{D}_{G/P}^{\lambda+\rho(\mathfrak{u})})$$

Quantizations of nilpotent covers

Proposition (Losev-MB-Matvieievskyi)

For each $\mathcal{A}_\lambda(\widetilde{\mathbb{O}}) \in Q(\widetilde{\mathbb{O}})$, there is a *unique* quantum co-moment map

$$\Phi : U(\mathfrak{g}) \to \mathcal{A}_\lambda(\widetilde{\mathbb{O}})$$

such that $\Phi|_{\mathfrak{z}(\mathfrak{g})} = 0$. The map Φ turns $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ into a finite-length, spherical Harish-Chandra bimodule for $U(\mathfrak{g})$. Write

$$I_\lambda(\widetilde{\mathbb{O}}) := \ker(\Phi).$$

This is a completely prime, primitive ideal.

Definition (Losev-MB-Matvieievskyi)

The *unipotent ideal* attached to $\widetilde{\mathbb{O}}$ is $I_0(\widetilde{\mathbb{O}})$.

Infinitesimal characters of quantizations of nilpotent covers

Take $\widetilde{\mathbb{O}} \in \mathrm{Cov}(G)$ corresponding to $(L, \widetilde{\mathbb{O}}_L) \in \Psi(G)$.

For each $\mathcal{A}_\lambda(\widetilde{\mathbb{O}}) \in Q(\widetilde{\mathbb{O}})$, write

$$\gamma_\lambda(\widetilde{\mathbb{O}}) = \text{infl char of } I_\lambda(\widetilde{\mathbb{O}}) \ \in \ \mathfrak{h}^*/W.$$

Lemma (Losev-MB-Matvieievskyi)

$\gamma_\lambda(\widetilde{\mathbb{O}}) = \gamma_0(\widetilde{\mathbb{O}}_L) + \lambda.$

This reduces the calculation of $\gamma_\lambda(\widetilde{\mathbb{O}})$ to the calculation of $\gamma_0(\widetilde{\mathbb{O}})$ for birationally rigid covers. The latter calculation was carried out in Losev-MB-Matvieievksyi (classical groups) and MB-Matvieievskyi (spin and exceptional groups).

Simple quantizations of nilpotent covers

When is $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ a simple algebra?

Theorem (Losev-MB-Matvieievskyi)

- The algebra $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ is simple if and only if the ideal $I_\lambda(\widetilde{\mathbb{O}})$ is maximal.
- The ideal $I_\lambda(\widetilde{\mathbb{O}})$ is maximal if and only if $\gamma_\lambda(\widetilde{\mathbb{O}})$ satisfies a straightforward combinatorial condition.
- This combinatorial condition is satisfied in an open subset of $\mathfrak{h}(\widetilde{\mathbb{O}})$ (including 0).

Examples later...

Real structures on quantizations of nilpotent covers

- Let σ be a compact form of $\mathfrak{g}$.
- If $\mathbb{O}$ is a nilpotent orbit, then σ preserves $\mathbb{O}$, induces a real form σ on $\mathbb{C}[\mathbb{O}]$.
- A cover $\widetilde{\mathbb{O}}$ is said to be *relevant* if it is birationally induced from a nilpotent orbit.
- If $\widetilde{\mathbb{O}}$ is relevant, then σ induces a real form σ on $\mathbb{C}[\widetilde{\mathbb{O}}]$.
- A quantization $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ of a relevant cover is *real* if σ lifts to a (necessarily unique) real form on $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$.
- If $\widetilde{\mathbb{O}}$ is relevant, then $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ is real if and only if

$$-\bar{\lambda} \in W(\widetilde{\mathbb{O}})\lambda$$

Hermitian bimodules for real quantizations of nilpotent covers

Let $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ be a real quantization of a relevant cover and let V be a Harish-Chandra $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$-bimodule.

- A Hermitian form $\langle \ , \ \rangle$ on V is *invariant* if

$$\langle xvy, w\rangle = \langle v, \sigma(y)w\sigma(x)\rangle, \qquad x, y \in \mathcal{A}_\lambda(\widetilde{\mathbb{O}}),\ v, w \in V.$$

- V is *Hermitian* if it admits a non-degenerate invariant Hermitian form.
- V is unitary if it admits a positive-definite invariant Hermitian form.
- If V is Hermitian/unitary as a $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ bimodule, it is Hermitian/unitary as a $U(\mathfrak{g})$-bimodule.
- If $\Phi : U(\mathfrak{g}) \to \mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ is *surjective*, then the converse is also true.

Hermitian quantizations of nilpotent covers

Let $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ be a real quantization of a relevant cover.

- $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ contains a *unique* copy of the trivial representation. Consider the projection

$$\eta : \mathcal{A}_\lambda(\widetilde{\mathbb{O}}) \to \mathbb{C}$$

- Define a Hermitian form on $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ by

$$\langle x, y \rangle := \eta(x\sigma(y))$$

Proposition

- $\langle \ , \ \rangle$ is invariant.
- $\langle \ , \ \rangle$ is the *unique* invariant Hermitian form on $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$.
- $\langle \ , \ \rangle$ is non-degenerate if and only if $\mathcal{A}_\lambda(\widetilde{\mathbb{O}})$ is simple.

Induction of quantizations of nilpotent covers

Suppose $\widetilde{\mathbb{O}}$ corresponds to $(L, \widetilde{\mathbb{O}}_L) \in \Psi(G)$. Choose a Levi subgroup $M \subset G$ containing L. Define

$$\widetilde{\mathbb{O}}_M := \mathrm{Bind}_L^M \widetilde{\mathbb{O}}_L \in \mathsf{Cov}(M).$$

Can define *parabolic induction* for filtered quantizations

$$\mathsf{Ind}_M^G : Q(\widetilde{\mathbb{O}}_M) \to Q(\widetilde{\mathbb{O}}).$$

Corresponds to the natural inclusion, on the level of parameters

$$\mathfrak{h}(\widetilde{\mathbb{O}}_M) = \mathfrak{z}(\mathfrak{l} \cap [\mathfrak{m}, \mathfrak{m}])^* \hookrightarrow \mathfrak{z}(\mathfrak{l} \cap [\mathfrak{g}, \mathfrak{g}])^* = \mathfrak{h}(\widetilde{\mathbb{O}}).$$

- If $\mathcal{A}_\lambda(\widetilde{\mathbb{O}}_M)$ is real, then $\mathsf{Ind}_M^G \mathcal{A}_\lambda(\widetilde{\mathbb{O}}_M)$ is real.
- If $\mathcal{A}_\lambda(\widetilde{\mathbb{O}}_M)$ is unitary, then $\mathsf{Ind}_M^G \mathcal{A}_\lambda(\widetilde{\mathbb{O}}_M)$ may not be Hermitian (i.e. simple), but if it is Hermitian, it is automatically unitary.

Complementary series for quantizations of nilpotent covers

Let $\widetilde{\mathbb{O}}$ be a relevant cover. Write:

$$\begin{aligned} &Q(\widetilde{\mathbb{O}}) \\ &\quad\cup \\ &Q_{\mathbb{R}}(\widetilde{\mathbb{O}}) = \{\text{real quantizations of } \mathbb{C}[\widetilde{\mathbb{O}}]\} \\ &\quad\cup \\ &Q_h(\widetilde{\mathbb{O}}) = \{\text{Hermitian quantizations of } \mathbb{C}[\widetilde{\mathbb{O}}]\} \\ &\quad\cup \\ &Q_u(\widetilde{\mathbb{O}}) = \{\text{unitary quantizations of } \mathbb{C}[\widetilde{\mathbb{O}}]\} \end{aligned}$$

Write $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$, $\mathfrak{h}_h(\widetilde{\mathbb{O}})$, $\mathfrak{h}_u(\widetilde{\mathbb{O}})$ for the corresponding parameter spaces. Recall

$$Q_h(\widetilde{\mathbb{O}}) = \{\mathcal{A} \in Q_{\mathbb{R}}(\widetilde{\mathbb{O}}) \mid \mathcal{A} \text{ simple}\}$$

Complementary series for quantizations of nilpotent covers

The set $\mathfrak{h}_h(\widetilde{\mathbb{O}})$ decomposes into connected components. If $S \subset \mathfrak{h}_h(\widetilde{\mathbb{O}})$, define

$$C(S) = \text{union of all connected components which meet } S \text{ nontrivially.}$$

This induces an operation on $Q_h(\widetilde{\mathbb{O}})$.

Proposition

If $S \subset Q_u(\widetilde{\mathbb{O}})$, then $C(S) \subset Q_u(\widetilde{\mathbb{O}})$.

Note: some quantizations in the family $C(S)$ may be reducible as $U(\mathfrak{g})$-bimodules. So $C(S)$ may *extend* the usual complementary series.

Conjectures

Conjecture

Suppose $\widetilde{\mathbb{O}}$ is relevant. Then

$$Q_u(\widetilde{\mathbb{O}}) = C(Q_h(\widetilde{\mathbb{O}}) \cap \bigcup_{M \supseteq L} \mathrm{Ind}_M^G Q_u(\widetilde{\mathbb{O}}_M)).$$

Conjecture

$$\Pi_{u,sph}(G) = \bigcup_{\widetilde{\mathbb{O}} \text{ relevant}} \{U(\mathfrak{g})/I_\lambda(\widetilde{\mathbb{O}}) \mid \mathcal{A}_\lambda(\widetilde{\mathbb{O}}) \in Q_u(\widetilde{\mathbb{O}})\}.$$

$\mathrm{Sp}(4, \mathbb{C})$: principal orbit

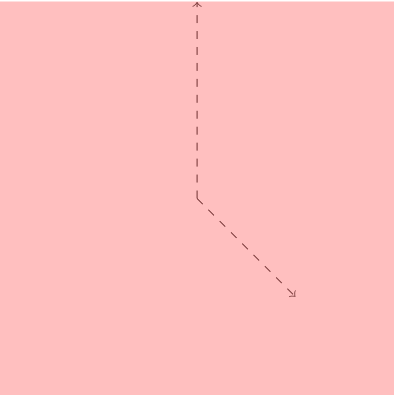


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: principal orbit

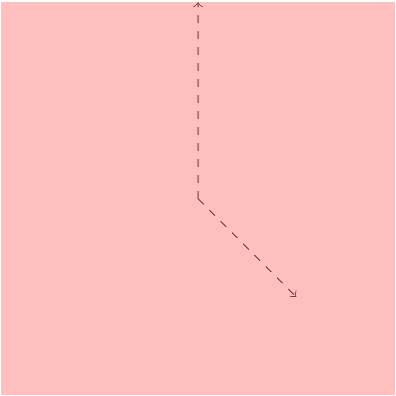


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: principal orbit

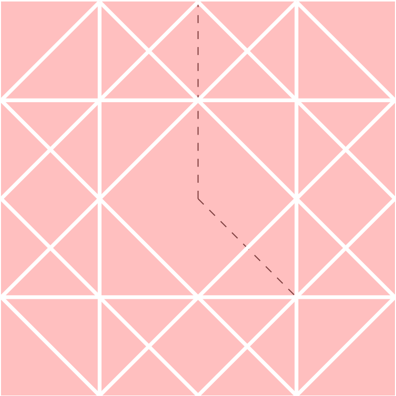
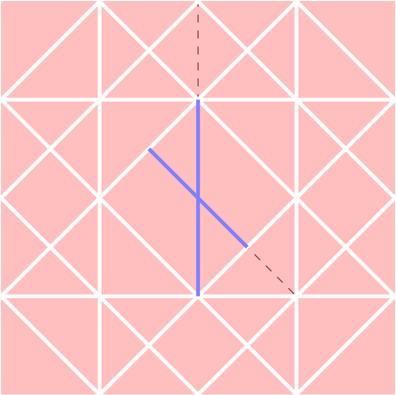


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: principal orbit



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$\mathrm{Sp}(4, \mathbb{C})$: principal orbit

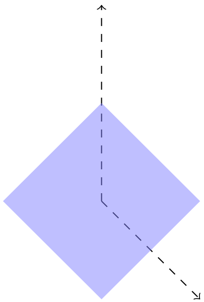

Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: subregular orbit

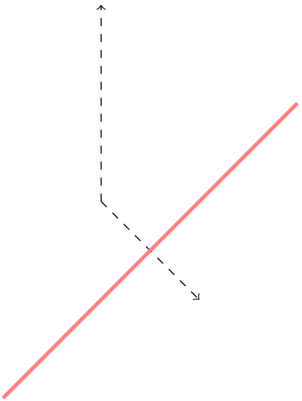

Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: subregular orbit

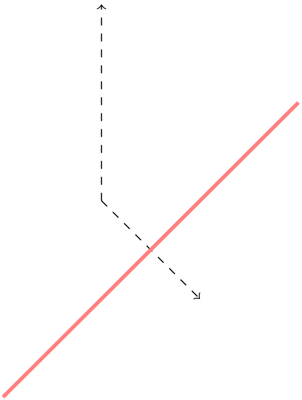

Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: subregular orbit

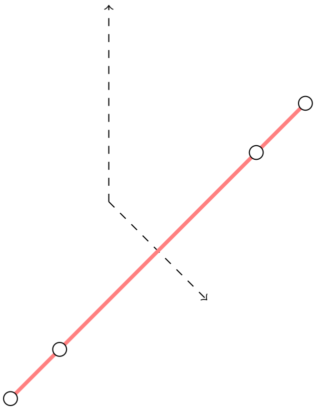

Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: subregular orbit

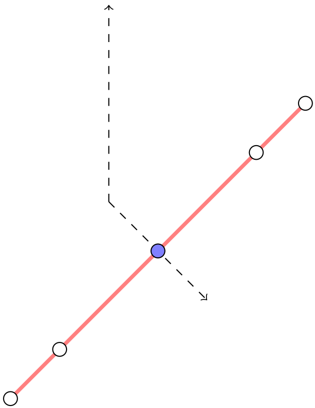

Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: subregular orbit

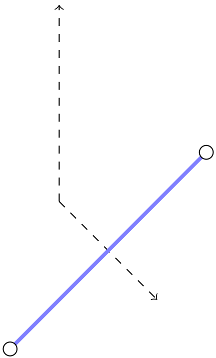

Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: double cover of subregular orbit

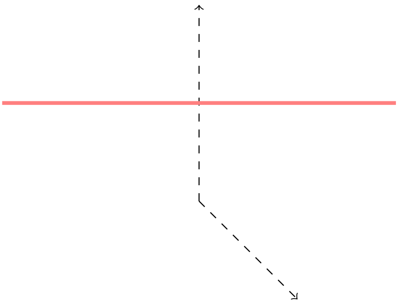

Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: double cover of subregular orbit

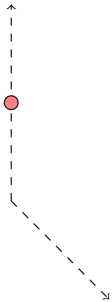

Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: double cover of subregular orbit

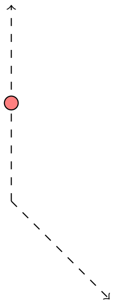

Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: double cover of subregular orbit

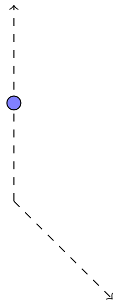

Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: minimal orbit

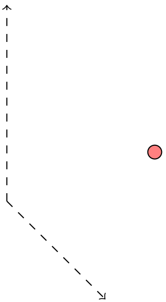

Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: minimal orbit

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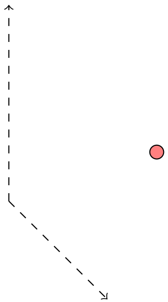

Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: minimal orbit

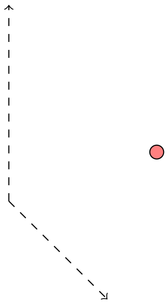

Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: minimal orbit

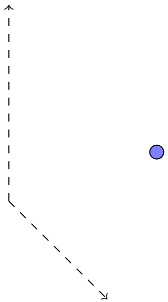

Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: zero orbit

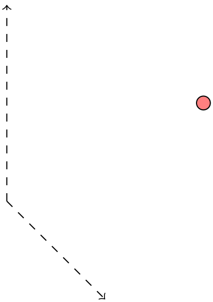

Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: zero orbit

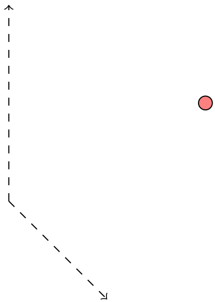

Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: zero orbit

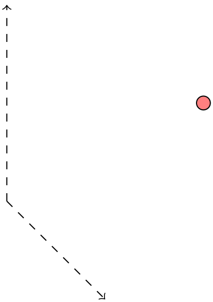

Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4,\mathbb{C})$: zero orbit

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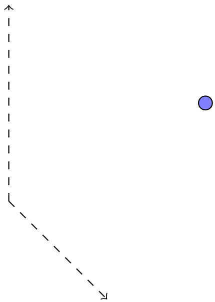
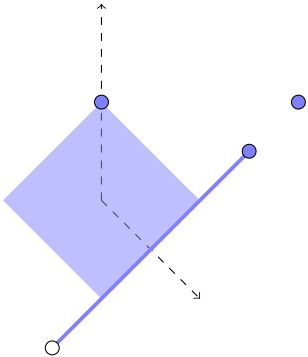
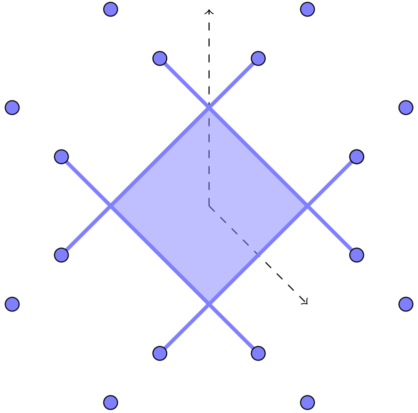


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$\mathrm{Sp}(4, \mathbb{C})$: putting it all together



$\mathrm{Sp}(4, \mathbb{C})$: putting it all together



$G_2(\mathbb{C})$: principal orbit

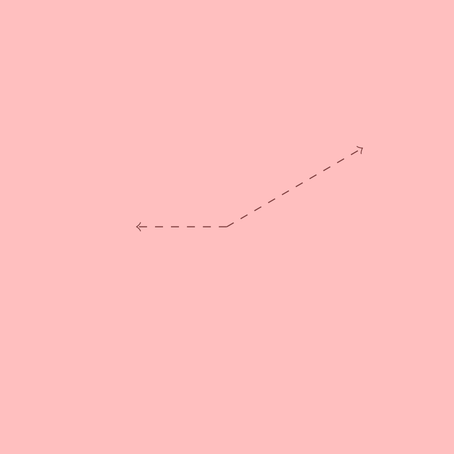


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: principal orbit

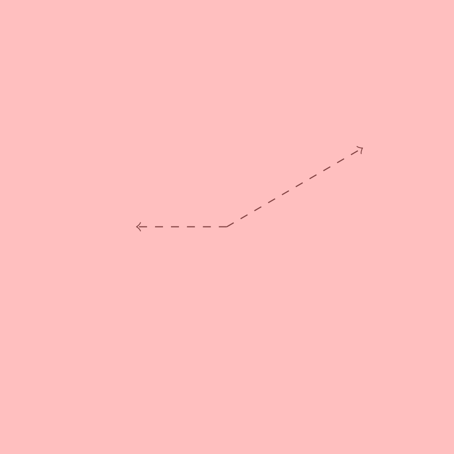


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: principal orbit

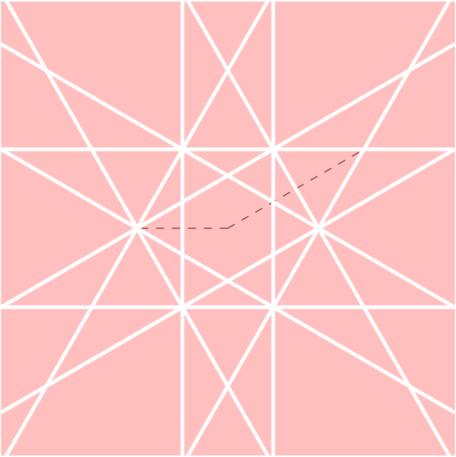


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: principal orbit

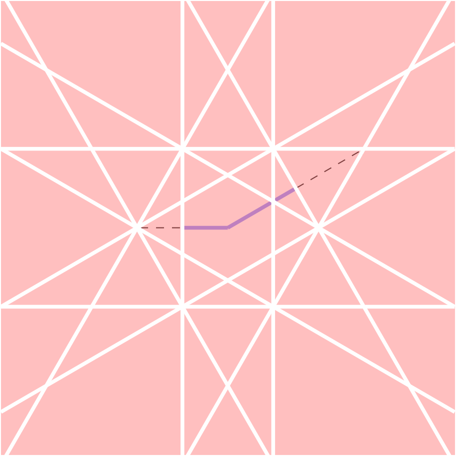


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: principal orbit

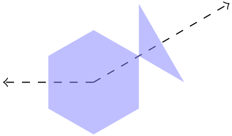


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: subregular orbit

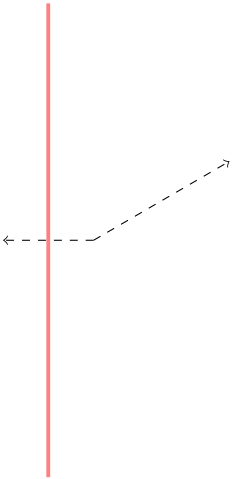


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: subregular orbit

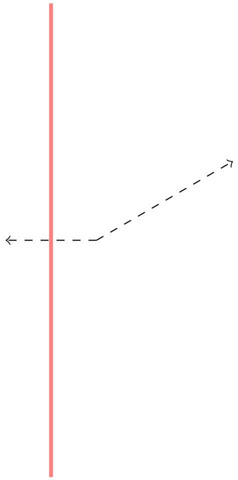


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: subregular orbit

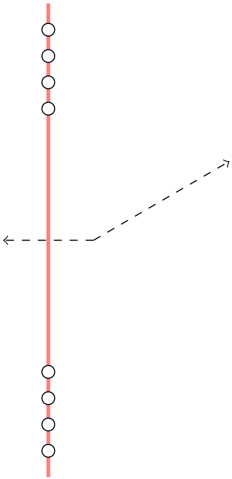


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: subregular orbit

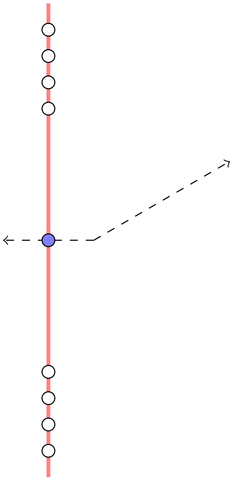


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: subregular orbit

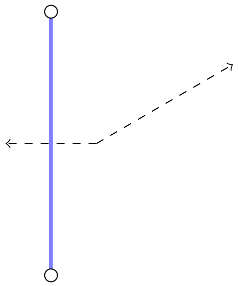


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: three-fold cover of subregular orbit

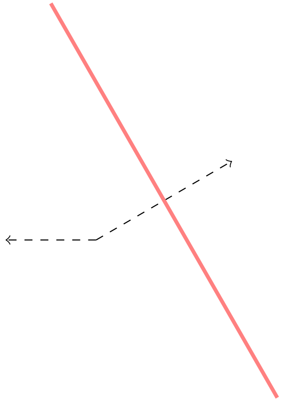


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: three-fold cover of subregular orbit

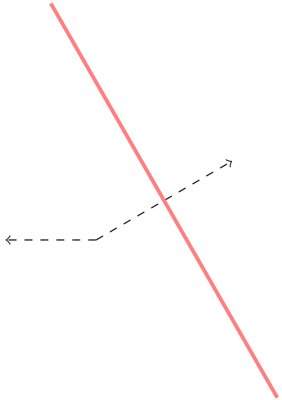


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: three-fold cover of subregular orbit

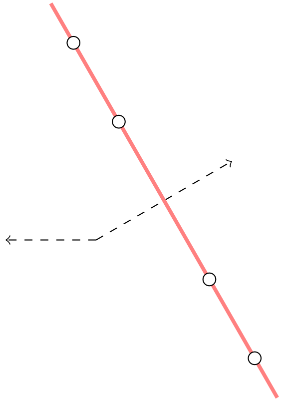


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: three-fold cover of subregular orbit

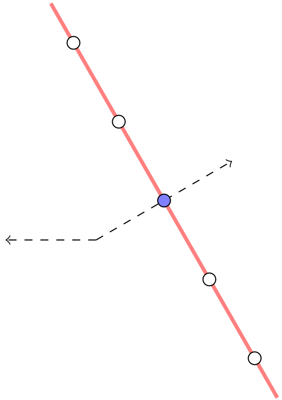


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: three-fold cover of subregular orbit

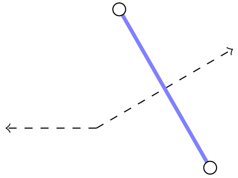


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: 8-dim rigid orbit

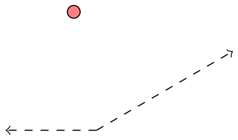


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: 8-dim rigid orbit

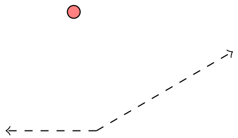


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: 8-dim rigid orbit

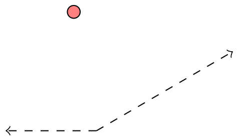


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: 8-dim rigid orbit

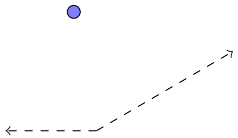


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: minimal orbit

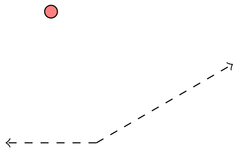


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: minimal orbit

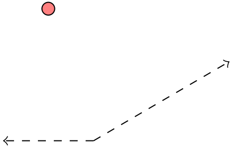


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: minimal orbit

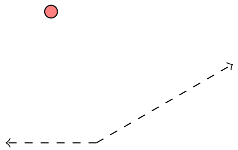


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: minimal orbit

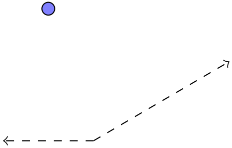


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: zero orbit

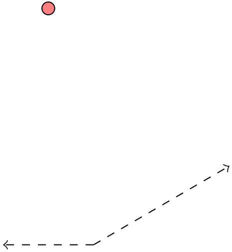


Figure: $\mathfrak{h}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: zero orbit

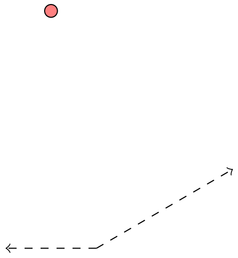


Figure: $\mathfrak{h}_{\mathbb{R}}(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: zero orbit

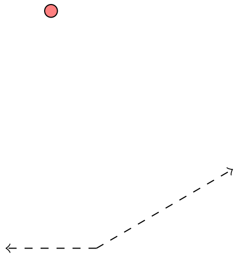


Figure: $\mathfrak{h}_h(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: zero orbit

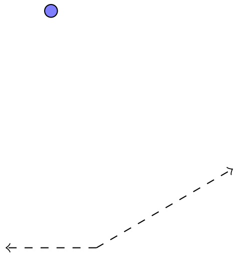
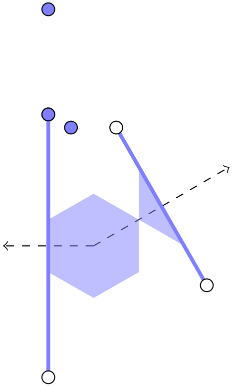


Figure: $\mathfrak{h}_u(\widetilde{\mathbb{O}})$

$G_2(\mathbb{C})$: putting it all together



$G_2(\mathbb{C})$: putting it all together