Quantization, the [orbit method, and](#page-20-0) unitary representations

#### David Vogan

<span id="page-0-0"></span>Quantization, the orbit method, and unitary representations

David Vogan

Department of Mathematics Massachusetts Institute of Technology

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### **Outline**

[Physics: a view from a neighboring galaxy](#page-2-0)

[Classical representation theory](#page-8-0)

[History of the orbit method in two slides](#page-10-0)

[Hyperbolic coadjoint orbits for reductive groups](#page-12-0)

[Elliptic coadjoint orbits for reductive groups](#page-16-0)

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<span id="page-2-0"></span>Physical system  $\longleftrightarrow$  complex Hilbert space H States  $\longleftrightarrow$  lines in  $H$ Observables ←→ linear operators {*A<sup>j</sup>* } on H Expected value of obs  $A \leftrightarrow \langle Av, v \rangle$ Energy ←→ special skew-adjoint operator *A*<sup>0</sup> Time evolution  $\longleftrightarrow$  unitary group  $t \mapsto \exp(tA_0)$ Observable *A* conserved  $\longleftrightarrow$   $[A_0, A] = 0$ Moral of the story: quantum mechanics is about Hilbert spaces and Lie algebras.

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<span id="page-10-0"></span>Unitary repn is Hilbert space  $\mathcal{H}_{\pi}$  with action  $G \times H_{\pi} \rightarrow H_{\pi}$ ,  $(q, v) \mapsto \pi(q)v$ respecting inner product:  $\langle v, w \rangle = \langle \pi(q) v, \pi(q) w \rangle$ .  $\pi$  is irreducible if has exactly two invt subspaces. Unitary dual problem: find  $G_{\mu}$  = unitary irreps of *G*.  $X \in \text{Lie}(G) \rightsquigarrow$  skew-adjoint operator  $d\pi(X)$ :

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One of Kostant's greatest contributions was understanding the power of the analogy

unitary repns Hilb space, Lie alg of ops  $\Leftrightarrow$ quantum mech systems Hilb space, Lie alg of ops

Unitary repns are hard, but quantum mech is hard too. How does an analogy help?

Physicists have a cheat sheet!

There is an easier version of quantum mechanics called classical mechanics. Theories related by



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 ${a, bc} = {a, b}c + b{a, c}$ 

*and a nondegeneracy condition.* Any smooth function *f* on *M* defines

Hamiltonian vector field  $\xi_f = \{f, \cdot\}.$ 

Example:  $M = \text{cotangent bundle.}$ Example:  $M =$ Kahler manifold.

Example:  $M = \text{conjugacy class of } n \times n \text{ matrices.}$ 

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**[Physics](#page-2-0)** 

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### **Pictures**

### Some conjugacy classes of  $2 \times 2$  real matrices



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[Physics](#page-2-0)

[Hyperbolic orbits](#page-12-0)



### **Pictures**

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### Representation theory and physics

### Here's how Kostant's analogy looks now.

unitary repns quantum mech system<br>Hilb space, Lie alg of ops Hilb space, Lie alg of ops

quantization ↑↓ classical

symplectic manifold

Hilb space, Lie alg of ops

limit quantization ↑↓ classical limit

# Hamiltonian *G*-space classical mech system<br>
symplectic manifold<br>
symplectic manifold

Poisson Lie alg of fns Poisson Lie alg of fns

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**[Representations](#page-8-0)** 

That is, the analogy suggests that there is a classical analogue of unitary representations.

Should make irreducible unitary correspond to

Must make sense of ↑↓. Physics ↑↓ not our problem.

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## *<sup>M</sup>* manifold with Poisson bracket {, } on smooth functions

 ${f, *}$   $\rightsquigarrow$   $\xi_f \in \text{Vect}(M)$ 

Hamiltonian vector field

*G* action on  $X \rightsquigarrow$  Lie alg hom  $q \rightarrow$  Vect(*M*),  $Y \mapsto \xi_Y$ . *M* is a Hamiltonian *G*-space if this Lie algebra map lifts

> *C* <sup>∞</sup>(*M*) % ↓ % ↓ g → Vect(*M*) *<sup>Y</sup>* <sup>→</sup> <sup>ξ</sup>*<sup>Y</sup> fY*

Map  $g \to C^{\infty}(M)$  same as moment map  $\mu \colon M \to g^*.$ 

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**[Representations](#page-8-0)** 

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> *C* <sup>∞</sup>(*M*)  $\nearrow$  $g \rightarrow \text{Vect}(M)$ *fY* % ↓ *<sup>Y</sup>* <sup>→</sup> <sup>ξ</sup>*<sup>Y</sup>*

Map  $g \to C^{\infty}(M)$  same as moment map  $\mu \colon M \to g^*.$ 

Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

**[Representations](#page-8-0)** 

*<sup>M</sup>* manifold with Poisson bracket {, } on smooth functions

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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Recall: Hamiltonian *G*-space *X* comes with (*G*-equivariant) moment map  $\mu: X \to g^*$ .

Kostant's theorem: homogeneous Hamiltonian  $G$ -space = covering of  $G$ -orbit on  $g^*$ .

Kostant's rep theory  $\leftrightarrow$  physics analogy now leads to Kirillov-Kostant philosophy of coadjt orbits:

 $\{$ irr unitary reps of  $G$ } =<sub>def</sub>  $\widehat{G}^{\{2\}}_{\longleftrightarrow} g^*/G$ . ( $\star$ )

**MORE PRECISELY. . .** restrict right side to "admissible" orbits (integrality cond). Expect to find "almost all" of *G*b: enough for interesting harmonic analysis.

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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# With the caveat about restricting to admissible orbits...  $\hat{G}$  ⇔ g<sup>\*</sup>/ $G$ . (\*)

 $(\star)$  true for *G* simply connected nilpotent (Kirillov) General idea  $(*)$ , without physics motivation, due to Kirillov. ( $\star$ ) true for *G* type I solvable (Auslander-Kostant).  $(\star)$  for algebraic *G* reduces to reductive *G* (Duflo). Case of reductive *G* is still open. Actually ( $\star$ ) is false for connected nonabelian reductive *G*.

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[Orbit method](#page-10-0)

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## Two ways to study representations for reductive *G*:

- 1. start with coadjt orbit, seek representation. Hard. 2. start with representation, seek coadjt orbit. Easy.
- Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)

Reductive Lie group  $G =$  closed subgp of  $GL(n,\mathbb{R})$ which is closed under transpose, and  $#G/G_0 < \infty$ .

From now on *G* is reductive.

Lie( $G$ ) =  $q \subset n \times n$  matrices. Bilinear form  $T(X, Y) = \text{tr}(XY) \Rightarrow g \stackrel{G\text{-eqvt}}{\simeq} g^*$ 

Orbits of  $G$  on  $g^* \subset$  conjugacy classes of matrices. Orbits of  $GL(n,\mathbb{R})$  on  $\mathfrak{g}^* =$  conj classes of matrices.

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[Hyperbolic orbits](#page-12-0)

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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[Hyperbolic orbits](#page-12-0)

# First example: hyperbolic orbits

 $G = GL(n, \mathbb{R})$ ,  $n = p + q$ ,  $x > y$  real numbers

 $O_{p,q}(X, Y) =_{\text{def}}$  diagonalizable matrices with eigvalues *<sup>x</sup>* (mult *<sup>p</sup>*) and *<sup>y</sup>* (mult *<sup>q</sup>*).

Define  $Gr(p, n) = Grassmann$  variety of *p*-dimensional subspaces of R *n* . <sup>O</sup>*p*,*<sup>q</sup>* is Hamiltonian *<sup>G</sup>*-space of dimension 2*pq*.  $O_{p,q}(x, y) \to Gr(p, n), \qquad \lambda \mapsto x$  eigenspace exhibits  $O_{p,q}(x, y)$  as affine bundle over  $Gr(p, n)$ 

General reductive *G*: *O* ⊂ g<sup>\*</sup> hyperbolic if elements are diagonalizable with real eigenvalues.

Always affine bundle over a compact real flag variety.

Quantization, the [orbit method, and](#page-0-0) unitary representations

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Define  $Gr(p, n) = Grassmann variety of$ *p*-dimensional subspaces of R *n* .

<sup>O</sup>*p*,*<sup>q</sup>* is Hamiltonian *<sup>G</sup>*-space of dimension 2*pq*.

 $O_{p,q}(x, y) \to Gr(p, n), \qquad \lambda \mapsto x$  eigenspace

exhibits  $O_{p,q}(x, y)$  as affine bundle over  $Gr(p, n)$ 

General reductive  $G: O \subset g^*$  hyperbolic if elements are diagonalizable with real eigenvalues.

Always affine bundle over a compact real flag variety.

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

Classical physics example:

configuration space  $X =$  manifold of positions.

State space  $T^*(X) = \frac{\text{symplectic manifold of}}{\text{positions and momenta}}$ .

 $\mathcal{H}=L^2(X)$ 

= square-integrable half-densities on *X*

 $=$  wave functions for quantum system.

Size of wave function  $\leftrightarrow$  probability of configuration. oscillation of wave function  $\leftrightarrow$  velocity.

Hamiltonian *G*-space *M* ≈  $T^*$ 

unitary representation  $\approx L^2(X) =$  square-integrable half-densities on *X*.

**KORK ERKERY EL ARA** 

Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

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configuration space  $X =$  manifold of positions. State space  $T^*(X) = \begin{cases} \text{symplectic manifold of} \ \text{of} \end{cases}$ 

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**Quantization** 

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\mathcal{H}=L^2(X)
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 $=$  square-integrable half-densities on  $X$  $=$  wave functions for quantum system.

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Hamiltonian *G*-space *M* ≈  $T^*$ unitary representation  $\approx L^2(X) =$  square-integrable half-densities on *X*.

Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

[Hyperbolic orbits](#page-12-0)

**KORK ERKERY EL ARA** 

Classical physics example:

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**Quantization** 

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### Size of wave function  $\leftrightarrow$  probability of configuration.

oscillation of wave function  $\leftrightarrow$  velocity.

Hamiltonian *G*-space *M* ≈  $T^*$ 

unitary representation  $\approx L^2(X) =$  square-integrable half-densities on *X*.

**KORK ERKERY EL ARA** 

Quantization, the [orbit method, and](#page-0-0) unitary representations

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Hamiltonian *G*-space *M* ≈  $T^*$ unitary representation  $\approx L^2(X) =$  square-integrable half-densities on *X*.**KORK ERKERY EL ARA** 

Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

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#### Kostant-Kirillov idea:

Hamiltonian *G*-space  $M \approx T^*(X)$   $\implies$ 

unitary representation  $\approx L^2(X) =$  square-integrable half-densities on *X*.

Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

[Hyperbolic orbits](#page-12-0)

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#### Two  $GL(n,\mathbb{R})$ -equivariant real line bdles on  $Gr(p, n)$ :

- 1.  $\mathcal{L}_1$ : fiber at *p*-diml  $S \subset \mathbb{R}^n$  is  $\bigwedge^p S;$
- 2.  $\mathcal{L}_2$ : fiber at *S* is  $\wedge^{n-p}(\mathbb{R}^n/S)$ .

Real numbers x and  $y \rightsquigarrow$  Hermitian line bundle

 $\mathcal{L}(x, y) = \mathcal{L}_1^{ix}$ 1 ⊗ L*iy* 2

Unitary representations of *GL*(*n*,R) associated to coadjoint orbits  $O_{p,q}(x, y)$  are

$$
\pi_{p,q}(x,y)=L^2(\mathrm{Gr}(p,n),\mathcal{L}(x,y)).
$$

Same techniques (still for reductive *G*) deal with all hyperbolic coadjoint orbits.

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

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Same techniques (still for reductive *G*) deal with all hyperbolic coadjoint orbits.

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

<span id="page-88-0"></span>Two *GL*(*n*,R)-equivariant real line bdles on Gr(*p*, *<sup>n</sup>*):

- 1.  $\mathcal{L}_1$ : fiber at *p*-diml  $S \subset \mathbb{R}^n$  is  $\bigwedge^p S$ ;
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Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

<span id="page-89-0"></span> $G = GL(2n, \mathbb{R}), x > 0$  real number

 $O_e(x) =_{def}$  real matrices  $\lambda$  with  $\lambda^2 = -x^2I$  $=$  diagonalizable  $\lambda$  with eigenvalues  $\pm xi$ .

 $O_e(x)$  is Hamiltonian *G*-space of dimension 2n<sup>2</sup>. Define a complex manifold

> $X=$  complex structures on  $\mathbb{R}^{2n}$  $\simeq$  *n*-dimensional complex subspaces  $S \subset \mathbb{C}^{2n}$  such that  $S + \overline{S} = \mathbb{C}^{2n}$

Last condition is open, so X open in  $Gr_{\mathbb{C}}(n, 2n)$ .  $O_e(x) \to X$ ,  $\lambda \mapsto ix$  eigenspace

is isomorphism  $O_e(X) \simeq X$ 

General reductive  $G: O \subset \mathfrak{g}^*$  elliptic if elements are diagonalizable with purely imaginary eigenvalues.

Always ≃ open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-88-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)[l](#page-15-0)[e](#page-16-0)[r.](#page-20-0)<br>استحقاقی دیگاه دیگاه دهان

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

### <span id="page-90-0"></span> $G = GL(2n, \mathbb{R}), x > 0$  real number

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Always ≃ open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-89-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)[l](#page-15-0)[e](#page-16-0)[r.](#page-20-0)<br>استحقاقی دیگاه دیگاه دهان

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

<span id="page-91-0"></span> $G = GL(2n, \mathbb{R}), x > 0$  real number

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Always ≃ open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-90-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)[l](#page-15-0)[e](#page-16-0)[r.](#page-20-0)<br>استحقاقی دیگاه دیگاه دهان

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

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A[l](#page-15-0)ways  $\simeq$  $\simeq$  $\simeq$  open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-91-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)le[r.](#page-20-0)<br>All the state state state is a completed on the state is the state state state is the state of the state state i

Quantization, the [orbit method, and](#page-0-0) unitary representations

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A[l](#page-15-0)ways  $\simeq$  $\simeq$  $\simeq$  open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-92-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)le[r.](#page-20-0)<br>All the state state state is a completed on the state is the state state state is the state of the state state i

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

<span id="page-94-0"></span> $G = GL(2n, \mathbb{R}), x > 0$  real number

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A[l](#page-15-0)ways  $\simeq$  $\simeq$  $\simeq$  open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-93-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)le[r.](#page-20-0)<br>All the state state state is a completed on the state is the state state state is the state of the state state i

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

<span id="page-95-0"></span>
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Always ' open orbit *X* on cplx fl[ag](#page-94-0) [va](#page-20-0)[r](#page-19-0)[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)[l](#page-15-0)[e](#page-16-0)[r.](#page-20-0)

Quantization, the [orbit method, and](#page-0-0) unitary representations

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Always ~ open o[r](#page-19-0)bit *X* on cplx fl[ag](#page-95-0) [va](#page-20-0)r[iety](#page-20-0)[:](#page-15-0) [K](#page-16-0)[äh](#page-20-0)[l](#page-15-0)[e](#page-16-0)[r.](#page-20-0)<br>All the setting and the setting

Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

First, complex manifold X: real space  $T_xX$  has complex structure  $j_x$ : real linear aut,  $j_x^2 = -I$ .

Second, symplectic:  $T_{\rm x}X$  has symp form  $\omega_{\rm x}$ .

Third, structures compatible:  $\omega_x(j_xu, j_xv) = \omega_x(u, v)$ .

These structures define indefinite Riemannian structure  $g_x(u, v) = \omega_x(u, j_xv)$ .

Kähler structure is positive if all *g<sup>x</sup>* are positive; signature (*p*, *<sup>q</sup>*) if all *<sup>g</sup><sup>x</sup>* have signature (*p*, *<sup>q</sup>*)

Example:  $X =$  complex structures on  $\mathbb{R}^{2n}$  has signature  $\left(\binom{n}{2},\binom{n+1}{2}\right)$  $\left(\begin{array}{c|c} 2 & 1 \end{array}\right)$  ((2)  $\binom{n+1}{2}$  or  $\left(\binom{n+1}{2}, \binom{n}{2}\right)$ 

Positive Kähler structures are better, but here we can't have them. Need direction. . .

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Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan

First, complex manifold X: real space  $T_xX$  has complex structure  $j_x$ : real linear aut,  $j_x^2 = -I$ . Second, symplectic:  $T_{\rm x}X$  has symp form  $\omega_{\rm x}$ . Third, structures compatible:  $\omega_x(j_xu, j_xv) = \omega_x(u, v)$ . These structures define indefinite Riemannian structure  $g_x(u, v) = \omega_x(u, j_x v)$ .

Kähler structure is positive if all *g<sup>x</sup>* are positive; signature  $(p, q)$  if all  $q_x$  have signature  $(p, q)$ 

Example:  $X =$  complex structures on  $\mathbb{R}^{2n}$  has signature  $\left(\binom{n}{2},\binom{n+1}{2}\right)$  $\left(\begin{array}{c|c} 2 & 1 \end{array}\right)$  ((2)  $\binom{n+1}{2}$  or  $\left(\binom{n+1}{2}, \binom{n}{2}\right)$ 

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#### Example: *<sup>U</sup>*(*n*)/*U*(1) *<sup>n</sup>* has *n*! equivariant Kähler structures. Here's how. . .

1. Distinct reals  $\ell = (\ell_1, \ldots, \ell_n) \rightsquigarrow U(n)$  coadjt orbit

 $O_e(\ell) = U(n) \cdot \text{diag}(i\ell_1, \ldots, i\ell_n);$ 

with natural symplectic structure.

2. Isomorphic to complex  $X =$  complete flags in  $\mathbb{C}^n$  by

 $\lambda \in O_{\mathbf{e}}(\ell) \mapsto \left( \{0\} \subset \mathbb{C}^n_{i\ell_m}(\lambda) \subset \mathbb{C}^n_{i\ell_n}(\lambda) + \mathbb{C}^n_{i\ell_{n-1}}(\lambda) \subset \cdots \right);$ 

here  $\mathbb{C}_{i\ell_j}^n(\lambda) =$  (one-diml)  $i\ell_j$ -eigenspace of  $\lambda$ .

3. Define  $\sigma$  = permutation putting  $\ell$  in decreasing order.<br>4. Jeannembians with  $X = \{z \in \mathbb{R}^n : z \in \mathbb{R}^n$ 

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Quantization, the [orbit method, and](#page-0-0) unitary representations

#### David Vogan
# Dealing with indefinite Kähler

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- *i*`*j* 3. Define  $\sigma$  = permutation putting  $\ell$  in decreasing order.<br>4. Jeographism with  $X = \lvert G \rvert$  is developed in a signature.
- 4. Isomorphism with  $X \rightsquigarrow$  Kähler structure of signature  $\left(\binom{n}{2}-\ell(\sigma),\ell(\sigma)\right).$

## David Vogan

[Elliptic orbits](#page-16-0)

**A DIA K B A SA A SA A SA A SA A SA SA SA BA** 

# How do you quantize a Kähler manifold?

# Kostant-Auslander idea:

Hamiltonian *G*-space *X* positive Kähler unitary representation  $= L^2$  holomorphic sections of holomorphic line bdle on *X*

But Kähler structures on

 $O_e(x) = 2n \times 2n$  real λ,  $λ^2 = -x^2$ 

are both indefinite.

New idea comes from Borel-Weil-Bott theorem about compact groups (proved algebraically by Kostant)

Quantization, the [orbit method, and](#page-0-0) unitary representations

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Quantization, the [orbit method, and](#page-0-0) unitary representations

David Vogan

# <span id="page-112-0"></span>Quantizing  $U(n) \cdot diag(i\ell_1, \ldots, i\ell_n) \subset u(n)^*$ <br>
• distinct real now equation  $\ell_1$  (n, 1)(2) (mod  $\mathbb{Z}$ ) E  $\ell_j$  distinct real; now assume  $\ell_j \equiv (n - 1)/2$  (mod Z). Put

$$
\rho=((n-1)/2,(n-3)/2,\ldots,(-n+1)/2)\in\mathbb{R}^n
$$

$$
\ell - \rho = (\ell_1 - (n-1)/2, \ldots, \ell_n + (n-1)/2) \in \mathbb{Z}^n.
$$

Get  $\mathcal{L}_{\ell-\rho}$  hol line bdle on  $X = \text{flags in } \mathbb{C}^n$ . Recall  $\sigma \cdot \ell$  decr; so  $\mu = \sigma \ell - \rho =$  dom wt for *U*(*n*). Write  $E_u = \text{irr rep of } U(n)$  of highest weight  $\mu$ .

Moral of the story: look for representations not in holomorphic sections, but in cohomological degree given by signature of Kähler [me](#page-111-0)[tri](#page-20-0)[c](#page-19-0)[.](#page-20-0)

Quantization, the [orbit method, and](#page-0-0) unitary representations

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[Elliptic orbits](#page-16-0)

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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# Theorem (Borel-Weil-Bott-Kostant) *Write*  $O_{\ell-p}$  = *sheaf of germs of hol secs of*  $\mathcal{L}_{\ell-p}$ *. Then*  $H^p(X, O_{\ell-p}) =$  $\left\{\right.$  $\overline{\mathcal{L}}$  $E_\mu$   $\rho = \ell(\sigma)$ <br>0 otherwise.

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Quantization, the [orbit method, and](#page-0-0) unitary representations

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