The Capelli eigenvalue problem

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- Key idea: $Mat_{n \times n} \longrightarrow$ Jordan algebra N; det $\longrightarrow$ Jordan norm φ

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- Proof involves Laplace transform on the tube domain $N + iZ$.

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- Hence to explicit Hilbert spaces for unitary subquotients of $I_P(s)$ [S. 92, INVENT. 101] , [Dvorsky-S. 98-99], SELECTA 4, INVENT. 138

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Theorem ([S. 1994, Prog Math 123] Kostant 65th Birthday)

For $\mu \in \Lambda_n$ there is a unique symm. polynomial $R_\mu^{(\rho)}(x) \in \mathbb{F}[x_1, \ldots, x_n]^{S_n}$ of degree $|\mu|$ such that $R_\mu^{(\rho)}(\lambda + \rho) = \delta_{\lambda\mu}$ for $|\lambda| \leq |\mu|$.

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Theorem ([*ibid.*])

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- (2) is a limit of (1) and implies (3) by work of Debiard-Sekiguchi.

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- $\mathbb{F} = \mathbb{Q}(q,t)$, S_n acts on $\mathbb{Z}^n$ and $\mathbb{F}[x_1,\ldots,x_n]$, generators $s_1,\ldots s_{n-1}$
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- The same approach proves Macdonald's conjectures for arbitrary root systems [Cherednik 1997, Selecta 3], and also for "type BC" Koorwinder polynomials [S. 1999, Annals 150]

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- Generalizes [S. 2013, Rep Theory 17] and [S.-Zhang 2017, IMRN]

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Theorem ([S.- Zhang 2018, Math Res Lett, to appear])

We have $\eta(L_\lambda) = P_\lambda^{\tau,\alpha}$ with τ and α as above.

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