Symmetric pairs and branching laws

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Representation theory, geometry, and quantization:
the mathematical legacy of Bertram Kostant
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Branching laws

$H \subset K$: compact connected Lie groups

$\widehat{K}$ and $\widehat{H}$: set of irreducible representations of K and H

$R(K)$ and $R(H)$: representation rings of K and H

Fundamental question

Understand the restriction morphism $R(K) \to R(H)$

Branching laws

The multiplicity map $m : \widehat{K} \times \widehat{H} \to \mathbb{N}$ is defined by

$$V_\lambda^K = \sum_{\mu \in \widehat{H}} m(\lambda, \mu) V_\mu^H, \quad \lambda \in \widehat{K}$$

Kostant multiplicity formula

Basic case : $H = T$ a maximal torus in K

Weyl character formula

$$\mathrm{Tr}(t \circlearrowright V_\lambda^K) = \sum_{w \in W} (-1)^w \frac{t^{w(\lambda+\rho)}}{\prod_{\alpha>0}(t^{\alpha/2} - t^{-\alpha/2})}, \quad t \in T$$

$R^{-\infty}(T) :=$ set of infinite sums $\sum_{\mu \in \widehat{T}} a_\mu \mathbb{C}_\mu$

T-module : $\mathfrak{k}/\mathfrak{t} \simeq \sum_{\alpha>0} \mathbb{C}_\alpha$

Kostant multiplicity formula

Identity in $R^{-\infty}(T)$:

$$V_\lambda^K|_T = \sum_{w \in W} (-1)^w \, \mathbb{C}_{w \cdot \lambda} \otimes \mathrm{Sym}(\overline{\mathfrak{k}/\mathfrak{t}})$$

Geometric realizations of V_λ^K

Flag variety : $\mathcal{F} := K/T$

Line bundle : $\mathcal{L}_\lambda \to K/T$ associated to $\lambda \in \widehat{T}$

Borel-Weil

$$V_\lambda^K = H^0(\mathcal{F}, \mathcal{L}_\lambda)$$

Dolbeault-Dirac operator :

$$D_\lambda : \mathcal{C}^\infty(\mathcal{F}, \wedge^+ T^{0,1}M \otimes \mathcal{L}_\lambda) \longrightarrow \mathcal{C}^\infty(\mathcal{F}, \wedge^- T^{0,1}M \otimes \mathcal{L}_\lambda)$$

Dolbeault theorem & Kodaira vanishing theorem

The operator D_λ is elliptic and

$$\begin{aligned} \mathrm{Index}^K(D_\lambda) &= \sum_j (-1)^j H^j(\mathcal{F}, \mathcal{L}_\lambda) \\ &= H^0(\mathcal{F}, \mathcal{L}_\lambda) \end{aligned}$$

Atiyah-Segal-Singer abelian localization

Take $r \in \mathfrak{t}$ regular. Vector field on $\mathcal{F}$: $\mathbf{r}(x) = r \cdot x$

Basic fact : $\{\mathbf{r} = 0\} = \mathcal{F}^T \simeq W$

Deformation idea (Atiyah-Singer, LNM 401,1974)

- Elliptic operator $D_\lambda \rightsquigarrow$ transversally elliptic operator $D^{\mathbf{r}}_\lambda$
- Abelian localization :

$$\begin{aligned} V^G_\lambda|_T &= \mathrm{Index}^T(D_\lambda) \\ &= \mathrm{Index}^T(D^{\mathbf{r}}_\lambda) \\ &= \sum_{w \in W} \mathrm{Index}^T(D^{\mathbf{r}}_\lambda|_{U_w}) \end{aligned}$$

- Direct computation :

$$\mathrm{Index}^T(D^{\mathbf{r}}_\lambda|_{U_w}) = (-1)^w \, \mathbb{C}_{w.\lambda} \otimes \mathrm{Sym}(\overline{\mathfrak{k}/\mathfrak{t}})$$

Symmetric pairs

Let $\sigma : K \to K$ be an involution. Let $K^\sigma := \{k \,|\, \sigma(k) = k\}$.

Consider a subgroup $H \subset K$ such that $(K^\sigma)_0 \subset H \subset K^\sigma$.

Theorem (Aomoto, Wolf, Matsuki, Rossman ...)

The set $H_{\mathbb{C}} \backslash \mathcal{F}$ is finite.

Critical set : $Z_\sigma := \{x \in \mathcal{F} \,|\, \sigma(K_x) = K_x\}$

Theorem revisited

- $H \backslash Z_\sigma$ is finite
- $\mathcal{O} \mapsto \mathcal{O} \cap Z_\sigma$ defines a bijective map $H_{\mathbb{C}} \backslash \mathcal{F} \xrightarrow{\sim} H \backslash Z_\sigma$

Geometric proof of Uzawa (1990)

Let $r \in \mathfrak{t}$ be regular. We consider the moment map

$$\Phi_r : \mathcal{F} \to (\mathfrak{k}^\sigma)^*.$$

- $Z(\Phi_r) =$ the critical set of $\|\Phi_r\|^2$
- $\varphi_t =$ flow generated by the gradient vector field $-\nabla\|\Phi_r\|^2$.
- $\pi(x) = \lim_{t\to\infty} \varphi_t(x)$ defines a H-equivariant projection

$$\pi : \mathcal{F} \to Z(\Phi_r).$$

Uzawa theorem

- $Z(\Phi_r) = Z_\sigma$ for any regular element r
- $\mathcal{P} \mapsto \pi^{-1}(\mathcal{P})$ defines a bijective map $H\backslash Z_\sigma \xrightarrow{\sim} H_{\mathbb{C}}\backslash\mathcal{F}$

Non abelian localization

Consider the Kirwan vector field

$$\kappa_r(x) = \Phi_r(x) \cdot x$$

- κ_r is the Hamiltonian vector field of $\frac{1}{2}\|\Phi_r\|^2$
- $\{\kappa_r = 0\} = Z_\sigma$

Deformation again

- Elliptic operator $D_\lambda \rightsquigarrow$ transversally elliptic operator $D_\lambda^{\kappa_r}$
- Non abelian localization :

$$\begin{aligned} V_\lambda^K|_H &= \mathrm{Index}^H(D_\lambda) \\ &= \mathrm{Index}^H(D_\lambda^{\kappa_r}) \\ &= \sum_{Hx \subset Z_\sigma} \mathrm{Index}^H(D_\lambda^{\kappa_r}|_{U_x}) \end{aligned}$$

Computation of $\mathrm{Index}^H(D_\lambda^{\kappa r}|_{U_x})$

Basic fact

$Q_x(\lambda) := \mathrm{Index}^H(D_\lambda^{\kappa r}|_{U_x})$ does not depends on r.

- $\mathfrak{R}_x \subset \mathfrak{k}_x^*$: roots relative to the Cartan subalgebra $\mathfrak{k}_x \circlearrowright \mathfrak{k} \otimes \mathbb{C}$.
- $\alpha \in \mathfrak{R}_x$ is an imaginary root if $\sigma(\alpha) = \alpha$.
- α is a compact imaginary root if $\sigma = Id$ on $(\mathfrak{k} \otimes \mathbb{C})_\alpha$
- α is a non-compact imaginary root if $\sigma = -Id$ on $(\mathfrak{k} \otimes \mathbb{C})_\alpha$

K_x-modules

$$\mathbb{E}_x^{\text{ci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha \quad \text{and} \quad \mathbb{E}_x^{\text{nci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha$$

Computation of $Q_x(\lambda)$

- Character $\mathbb{C}_{\gamma(x)}$ of the abelian group H_x:

$$\gamma_x := \lambda_x + \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \cap \sigma(\mathfrak{R}_x^+) \\ \sigma(\alpha) \neq \alpha}} \alpha + \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \alpha$$

- Let $m_x = \frac{1}{2} |\mathfrak{R}_x^+ \cap \sigma(\mathfrak{R}_x^+) \cap \{\sigma(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\mathrm{nci}}$

Theorem, P. (2018)

We have the decomposition

$$V_\lambda^K|_H = \sum_{Hx \in H \backslash Z_\sigma} Q_x(\lambda), \quad \text{with}$$

$$Q_x(\lambda) = (-1)^{m_x} \mathrm{Ind}_{H_x}^{H} \left(\mathbb{C}_{\gamma_x} \otimes \mathrm{Sym}(\mathbb{E}_x^{\mathrm{nci}}) \otimes \bigwedge \mathbb{E}_x^{\mathrm{ci}} \right)$$

Example : $U(1) \subset SU(2)$

- $\mathcal{F} \simeq \mathbb{S}^2$ and $Z_\sigma = \{S, N, E\}$
- For $Hx = E$: $H_x \simeq \mathbb{Z}_2$, $\mathbb{E}_x^{\mathrm{nci}} = \mathbb{E}_x^{\mathrm{ci}} = \{0\}$, $\mathbb{C}_{\gamma_x} = \mathbb{C}_n|_{\mathbb{Z}_2}$.
- For $Hx = N$: $H_x = T$, $\mathbb{E}_x^{\mathrm{nci}} = \mathbb{C}_2$, $\mathbb{E}_x^{\mathrm{ci}} = \{0\}$, $\mathbb{C}_{\gamma_x} = \mathbb{C}_{n+2}$.
- For $Hx = S$: $H_x = T$, $\mathbb{E}_x^{\mathrm{nci}} = \mathbb{C}_{-2}$, $\mathbb{E}_x^{\mathrm{ci}} = \{0\}$, $\mathbb{C}_{\gamma_x} = \mathbb{C}_{-n-2}$.

Final decomposition

$$
\begin{aligned}
V_n|_T &= \mathbb{C}_n \otimes \sum_{k\in\mathbb{Z}} \mathbb{C}_{2k} - \mathbb{C}_{n+2} \otimes \mathrm{Sym}(\mathbb{C}_2) - \mathbb{C}_{-n-2} \otimes \mathrm{Sym}(\mathbb{C}_{-2}) \\
&= \sum_{k=-n}^{0} \mathbb{C}_{2k+n}.
\end{aligned}
$$

Example : $H \subset H \times H$

- $\mathcal{F} = H/T \times H/T$ and $Z_\sigma = \bigcup_{w \in W_H} Z_w$ with $Z_w = H \cdot (w, 1)$.
- $x = (w, 1)$: $H_x = T$, $\mathbb{E}_x^{\mathrm{nci}} = \mathbb{E}_x^{\mathrm{ci}} = \{0\}$, $\mathbb{C}_{\gamma_x} = \mathbb{C}_{w(\lambda+\rho)+\mu+\rho}$

Final decomposition

$$V_\lambda^H \otimes V_\mu^H = (-1)^{\dim(H/T)/2} \sum_{w \in W_H} (-1)^w \mathrm{Ind}_T^H \left(\mathbb{C}_{w(\lambda+\rho)+\mu+\rho}\right)$$

- When $H = SU(2)$ and $m \geq n \geq 0$ we have

$$\begin{aligned} V_n \otimes V_m &= \mathrm{Ind}_{U(1)}^{SU(2)}(\mathbb{C}_{m-n}) - \mathrm{Ind}_{U(1)}^{SU(2)}(\mathbb{C}_{m+n+2}) \\ &= \sum_{k=0}^{n} \mathrm{Ind}_{U(1)}^{SU(2)}((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_{m+n-2k}) = \sum_{k=0}^{n} V_{m+n-2k} \end{aligned}$$

We recognize here the classical Clebsch-Gordan relations.

Example : $U(p) \times U(q) \subset U(p+q)$

- Let $K = U(p+q)$ with $p \geq q \geq 1$, and $H = U(p) \times U(q)$.
- Here $|H \backslash Z_\sigma / W| = q + 1$.
- For $j \in \{0, \ldots, q\}$ we consider the subgroups

$$H_j = U(p-j) \times U(q-j) \subset H, \text{ and } T_{p-j} \times T_{2j} \times T_{q-j} = T$$

Final decomposition

$$V_\lambda^{U(n)}|_{U(p)\times U(q)} = \sum_{j=0}^{q} \sum_{w \in W} \frac{\pm 1}{|W_j|} Q_j^w(\lambda) \otimes \chi^{\otimes j},$$

where $Q_j^w(\lambda)$ is equal to

$$\mathrm{Ind}_{H_j \times T_{2j}}^{H} \left(\mathrm{Hol}_{T_{p-j} \times T_{q-j}}^{H_j} \left(\mathbb{C}_{\gamma_{j,w}^1} \otimes \mathrm{Sym}(\mathbb{M}_j^w) \right) \otimes \mathbb{C}_{\gamma_{j,w}^2} \otimes L^2(T_{2j}/T'_{2j}) \right)$$

Deformation of Dirac operators

Geometric package

- M oriented manifold of even dimension
- $\mathcal{S} =$ a spin-c structure on M
- a compact Lie group $H \circlearrowleft (M, \mathcal{S})$
- a H-equivariant vector field η on M
- Hypothesis on η : $\eta(m) \in T_m(H \cdot m)$ for all $m \in M$

• H-equivariant Dirac operator $D_{\mathcal{S}} : \mathcal{C}^\infty(M, \mathcal{S}^+) \longrightarrow \mathcal{C}^\infty(M, \mathcal{S}^-)$

• principal symbol of D : $\mathrm{cl}_{\mathcal{S}}$

Deformation of $\mathrm{cl}_{\mathcal{S}}$

$$\mathrm{cl}^{\eta}_{\mathcal{S}}|_m(v) := \mathrm{cl}_{\mathcal{S}}|_m(v + \eta(m))$$

Deformation of Dirac operators

- When M is compact :

$$\mathcal{Q}_H(M,\mathcal{S}) := \mathrm{Index}^H(D_{\mathcal{S}}) = \mathrm{Index}^H(D^\eta_{\mathcal{S}}) = \sum_{Z\subset\{\eta=0\}} \mathrm{Index}^H(D^\eta_{\mathcal{S}}|_{U_Z})$$

- When M is non compact but $\{\eta = 0\}$ is compact :

$$\ll \mathrm{Index}^H(D^\eta_{\mathcal{S}}) \gg := \sum_{Z\subset\{\eta=0\}} \mathrm{Index}^H(D^\eta_{\mathcal{S}}|_{U_Z})$$

Formal geometric quantization

$\eta_{\mathcal{S}}$: preferred choice associated to the line bundle $\det(\mathcal{S})$.

$$\mathcal{Q}_H^{-\infty}(M,\mathcal{S}) = \ll \mathrm{Index}^H(D^{\eta_{\mathcal{S}}}_{\mathcal{S}}) \gg$$

Application to discrete series representations

- G semi-simple real Lie group with an involution σ
- π^G_λ discrete series representation attached to $G\lambda$

Hypothesis

We suppose that the restriction $\pi^G_\lambda|_{G^\sigma}$ is admissible:

$$\pi^G_\lambda|_{G^\sigma} = \sum_\mu m(\lambda,\mu)\, \pi^{G^\sigma}_\mu$$

- the moment map $\Phi_\sigma : G\lambda \to (\mathfrak{g}^\sigma)^*$ is proper
- let $Y := \Phi_\sigma^{-1}((\mathfrak{k}^\sigma)^*)$: a smooth symplectic K^σ-manifold

Theorem, P. (2017)

$$\mathcal{Q}^{-\infty}_{\widetilde{K^\sigma}}(Y, \mathcal{S}_Y) = \sum_\mu m(\lambda,\mu)\, \pi^{\widetilde{K^\sigma}}_\mu$$

Thank you for your attention !