

Adapted Pairs and Polynomiality of Invariants. Conference in Memory of Bertram Kostant

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1. Theme

Square roots Bert missed and the troubles they cause.

2. Definitions and Goals

Let \mathfrak{g} be a semisimple Lie algebra.

Given a parabolic subalgebra \mathfrak{p} , let

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{r} \oplus \mathfrak{m}^-,$$

be the triangular decomposition it defines. In particular $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}$.
If \mathfrak{p} is the Borel \mathfrak{b} , then \mathfrak{n} will replace \mathfrak{m} and \mathfrak{h} will replace \mathfrak{r} .

One can study algebras of invariant functions given by this data.
Often they are polynomial but this (and the contrary) can be hard to ascertain.

3. An example of Dixmier.

Suppose $\mathfrak{g} = \mathfrak{sl}(n)$.

Then Dixmier observed that the minors meeting the upper right hand corner lying strictly above the diagonal generate $S(\mathfrak{n})^{\mathfrak{n}}$ as a polynomial algebra.

Incidentally something similar happens for $S(\mathfrak{m}^*)^{\mathfrak{n}}$, when $n = 2m$ is even and \mathfrak{t} is given by two blocks of size m . Then this algebra is polynomial, generated by the minors meeting the upper right hand corner of the *lower* block.

However the comparison ends there. Thus $S(\mathfrak{m})^n$ is always polynomial whilst $S(\mathfrak{m}^*)^n$ is not - for example in $\mathfrak{sl}(6)$ with τ defined by block sizes $(1, 2, 2, 1)$.

Polynomiality would follow if B had a dense orbit in \mathfrak{m} and this is a first example where that fails.

4. The Kostant Cascade.

Kostant considered the weights $\beta_1, \beta_2, \dots, \beta_t$ of vectors lying on the anti-diagonal above the diagonal. He noticed that the weights of Dixmier generators are $\beta_1, \beta_1 + \beta_2, \dots, \beta_1 + \dots + \beta_t$. Moreover this is a free set of generators of the semi-group of the set of all dominant weights lying in $\sum_{i=1}^t \mathbb{N}\beta_i$.

Remarkably he gave an abstract construction of the β_i , now called the Kostant cascade, which applied to all semi-simple Lie algebras and announced that $S(\mathfrak{n})^{\mathfrak{n}}$ is polynomial on generators with similarly defined weights.

Hearing of this in 1973, I gave a proof (much too nitty-gritty for Bert). The paper, with several other results, was published in 1977. It also showed that $S(\mathfrak{b})^n$ is polynomial on exactly rank g generators.

It took me another 24 years to pluck up the courage to ask Bert for his proof.

5. The Missing Square roots.

Kostant had elegantly used the Hopf dual to construct invariants. Then the generators come out with weights of the form $\varpi_i - w_0\varpi_i$, as ϖ_i runs over the fundamental weights.

By this time I had considered something similar in the quantum case and knew this gave the wrong answer!

The trouble is that answer should be $\varepsilon_i(\varpi_i - w_0\varpi_i)$ with $\varepsilon_i \in \{\frac{1}{2}, 1\}$. When $\varepsilon_i = \frac{1}{2}$ we call the corresponding simple root **intractable**.

In other words using the Hopf dual one must sometimes take a square root of the resulting invariant.

There are no intractable roots in types A, C . Otherwise they follow the Kostant cascade starting from the simple root attached to the highest root.

Remarkably they form strings of next nearest neighbours.

6. A Way Out?

There is an easy way out of the difficulty caused by the intractable roots.

By localisation one can easily show that the weights of weight vectors in the fraction field lie in the set $\sum_{i=1}^t \mathbb{Z}\beta_i$. This means that the required square root must lie in the fraction field, hence in the invariant algebra itself.

All very well; but the problem is not disposed of so fast!

7. Biparabolics

Suppose one replaces \mathfrak{b} by a parabolic (or even biparabolic) subalgebra \mathfrak{p} , and one wants to show that the Poisson semi-centre $Sy(\mathfrak{p})$ of $S(\mathfrak{p})$ is polynomial.

Then one can also get some invariants using the Hopf dual. This gives a lower bound on the character of the invariant (which is well-defined if \mathfrak{g} is simple and \mathfrak{p} is proper).

On the other by less elegant arguments using nilpotent action one can get an upper bound.

When these bounds coincide, $Sy(\mathfrak{p})$ is polynomial.

8. Failure of coincidence of bounds.

Briefly the bounds fail to coincide because certain crucial ε_j fail to equal 1.

Moreover for the Heisenberg parabolic in type E_8 , Yakimova showed using partly a computer computation that the invariant algebra is not polynomial.

Let \mathfrak{a} be an algebraic Lie algebra. Define the index $\ell(\mathfrak{a})$ of \mathfrak{a} to be the codimension of a regular co-adjoint orbit in \mathfrak{a}^* .

Even when bounds fail to coincide one can calculate the index of a biparabolic in terms of the Kostant cascade, confirming a conjecture of Tauvel and Yu.

9. A Degree Sum

Suppose that \mathfrak{a} is a biparabolic and the above bounds coincide. Then the sums of the degrees of homogeneous generators in $Sy(\mathfrak{p})$ is the “magic” number $c(\mathfrak{a}) := \frac{1}{2}(\dim \mathfrak{a} + \ell(\mathfrak{a}))$.

More generally let $p_{\mathfrak{a}}$ be the fundamental semi-invariant of \mathfrak{a} .

Motivated by our result, Ooms and Van den Bergh showed that if $S(\mathfrak{a})$ admits no proper semi-invariants and if the invariant algebra $Y(\mathfrak{a})$ were polynomial, then the sums of the degrees of the homogeneous generators equals $c(\mathfrak{a}) - \deg p_{\mathfrak{a}}$.

Their proof was a little technical, but following a very simple argument of Panyushev and in work with Shafrir, we showed it enough that $p_{\mathfrak{a}}$ is invariant and \mathfrak{a} is unimodular. This applies in particular to a centralizer Lie algebra \mathfrak{g}^x with \mathfrak{g} semisimple whose symmetric algebra can admit non-trivial semi-invariants.

10. Adapted Pairs and Sections.

The following was directly inspired by two famous papers of Kostant published in 1959 and 1963.

Let \mathfrak{a} be an algebraic Lie algebra. An adapted pair is a pair $h \in \mathfrak{a}$ semisimple and $\xi \in \mathfrak{a}_{reg}^*$ such that $(\text{ad } h)\xi = -\xi$. For \mathfrak{a} semisimple an adapted pair is provided by $\frac{2}{3}$ of a principal s -triple.

Adapted pairs are difficult to find but their construction is only linear algebra. One can ask if the existence of an adapted pair implies the polynomiality of the invariant algebra $Y(\mathfrak{a})$.

I found an adapted pair for all truncated biparabolics in type A .

This is especially difficult because we have no idea of the actual form of the invariants.

Also a similar result cannot hold outside type A .

The referee kindly pointed out that this will “weaken the overall *long-lasting* impact of the work” and promptly turned down the paper.

I drew succour from a comment of I. G. Macdonald. “We study semisimple Lie algebras by pretending not to know their classification”.

Given an adapted pair, let $V \subset \mathfrak{a}^*$ be an $\text{ad } h$ stable complement to $(\text{ad } h)\xi$ in \mathfrak{a}^* .

Then the restriction of $Y(\mathfrak{a})$ to $\mathbb{C}[\xi + V]$ is injective. Polynomiality of $Y(\mathfrak{a})$ will result if this restriction is also surjective.

In this case one calls $\xi + V$ a section. It meets almost every regular orbit at exactly one point and transversally. It meets every regular orbit if the nilcone is irreducible.

Inequivalently adapted pairs give different irreducible components of the nilcone. Many such exist in type A .

Yet suppose that p_{α} is an invariant and α is unimodular.

Then if $Y(\alpha)$ is polynomial, one may show that the degrees of the eigenvalues of $\text{ad } h$ on V are just the degrees minus one of the homogeneous generators.

The above also gives a way of proving non-polynomiality (for example if some eigenvalues are negative) and was intended to recover the Yakimova counter-example. Unfortunately no adapted pair was found!

Yet Moreau reported on another example of a biparabolic for which polynomiality can be denied using just this argument.

11. An improved upper bound

In some cases the existence of an adapted pair can give rise to an improved upper bound and if this coincides with the lower bound then polynomiality of $Y(\alpha)$ results.

Thus it was possible to show that $Sy(\mathfrak{p})$ for all Heisenberg parabolics for \mathfrak{g} simple outside type E_8 .

Actually in this case \mathfrak{p}' is a centralizer algebra and the result was obtained by a different method by Panyushev, Premet and Yakimova.

On the other hand this method was extended By Lamprou and Millet to show that $Sy(\mathfrak{p})$ is polynomial for all maximal parabolics in type B defined by the intractable roots. For the rest, bounds coincide.

12. The Letzter Map.

One can easily generalize the notion of an adapted pair to the case when the coadjoint module is replaced by an arbitrary finite dimensional module M .

This is particularly interesting when \mathfrak{a} is reductive as it is much more plausible that the existence of an adapted pair implies the polynomiality of invariants.

Indeed let $(h, m) \in \mathfrak{a} \times M_{reg}$ be an adapted pair, with V an h -stable complement to $\mathfrak{a}.m$ in M .

Now consider the restriction φ of $S(V^*)^{\mathfrak{a}}$ to $\mathbb{C}[\mathbb{C}m + V]^h$. It is a fortiori injective and its surjectivity is equivalent to the surjectivity of our previous map.

The advantage of φ is that it is linear and preserves degree.

Take $p \in \mathbb{C}[m + V]^h$ viewed as an element of $S(V^*)$. Clearly $U(\mathfrak{a})p/U(\mathfrak{a})_+p$ is the trivial module and has dimension ≤ 1 .

By reductivity, there exists $\theta(p) \in S(V^*)^{\mathfrak{a}}$ such that $p - \theta(p) \in U(\mathfrak{a})_+p$.

One checks that θ is linear. It clearly preserves degree.

Then for surjectivity it is enough to show that θ is injective.

In the co-adjoint case this would give another proof of the Chevalley theorem. At least in this case there is in principle a way to check injectivity using the trace.

13. Frobenius Lie Algebras

A Lie algebra \mathfrak{a} is said to be Frobenius if $\ell(\mathfrak{a}) = 0$.

It will be convenient to assume that \mathfrak{a} is algebraic. Then $S_{\mathfrak{y}}(\mathfrak{a}) = Y(\mathfrak{a}^t)$.

A simple general argument can be used to show this common algebra is factorial.

On the other hand $S_{\mathfrak{y}}(\mathfrak{a})$ is multiplicity-free as an \mathfrak{a} module.

Consequently $S_{\mathfrak{y}}(\mathfrak{a})$ is a polynomial algebra with generators being the irreducible weight vectors.

Ooms took this one step further and showed that every generator is a factor of p_α . On the other hand for a Frobenius Lie algebra p_α is easy to compute.

All fine and good but how does determine the irreducible factors?

Indeed

The obvious mathematical breakthrough would be to factor large prime numbers. Bill Gates

14. Frobenius Lie Algebras of biparabolic type

In the biparabolic case one does in principle know the weights of the irreducible factors, at least up to that pesky factor of 2.

To cut a rather long story short this is one case where one can show that the supposed square roots exist. (In general they do not).

If the ambient Lie algebra \mathfrak{g} (that is the one for which \mathfrak{a} is a biparabolic) is exceptional, then there is essentially one case to consider. It is also the only case that needs to be considered if \mathfrak{a} is parabolic.

Let π' be a subset of the simple roots π defining a parabolic subalgebra $\mathfrak{p}_{\pi'}$.

Let i_{π} (resp. $i_{\pi'}$) be the involution of π defined by the unique longest element of the Weyl group defined by π (resp. π').

Then $\mathfrak{p}_{\pi'}$ is Frobenius if and only if every $\langle i_{\pi} i_{\pi'} \rangle$ orbit meets $\pi \setminus \pi'$ at exactly one point.

Thus in type E_6 the parabolic subalgebra $\mathfrak{p}_{\pi'}$ is Frobenius when $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. In this case there are two $\langle i_{\pi} i_{\pi'} \rangle$ orbits.

The weights of the corresponding generating invariants are given by orbit sums. The one whose square root we have to find is of degree 8 in 50 variables. The second has degree 18.

Note that $8 + 18 = 26 = \frac{1}{2}(36 + 10 + 6) = \dim \mathfrak{p}_{\pi'}$, which is degree of the fundamental semi-invariant of $\mathfrak{p}_{\pi'}$.

One simply constructs an invariant of degree 4 of suitable weight using tensor product decomposition.

For biparabolics in classical type there are infinitely many cases to consider so we need a general theory.

I have only time to quote the referee who stated that “an ingenious induction was used”.

15. Meta-sections.

Take \mathfrak{g} simple.

One can ask if there is a result for which the pesky square roots are in full demand. For example in the existence of a section.

It turns out that $Sy(\mathfrak{b})$ and $Y(\mathfrak{n})$ have the same set of weights, multiplicity-free in the second case.

Thus one can construct the algebra

$$A(\mathfrak{g}) = \bigoplus Y(\mathfrak{n}^-)_{-\lambda} Sy(\mathfrak{b})_{\lambda},$$

where the sum is over the dominant weights lying in the span of the Kostant cascade.

With Millet we conjectured that $Y(\mathfrak{g})$ is recovered as the image of $A(\mathfrak{g})$ under the Letzter map. This we proved in types A, C , where pesky square roots do not appear.

But this was deleted from the article as the referee commented that

Ça ressemble à une plaisanterie de déduire la structure $Y(\mathfrak{g})$ de celle de $Sy(\mathfrak{b})$.

We gave this in another paper, which we wanted to call it “a joke in the theory of invariants”, but that did not go down too well.

Theorem

There exists a regular ad-nilpotent element $y \in \mathfrak{g}$ and a subspace V of \mathfrak{g} having dimension $\ell(\mathfrak{g})$ such that the restriction map induces an isomorphism of $A(\mathfrak{g})$ onto the algebra of regular functions on $y + V$.

This only works because the square roots dutifully turn out. It is not a section unless the elements of $Y(\mathfrak{n}^-)_{-\lambda}$ become non-zero scalars on restriction.

However one can turn it into a section for all types except those for which the penultimate element of the Kostant cascade is of type C_2 , that is in types $C, B(2m), F(4)$. Generally the construction of these sections through an adapted pair is not possible (for example in type $G(2)$).

16. Richardson's Theorem.

By Richardson's theorem $S(\mathfrak{m}^*)^{p'}$ is multiplicity free and hence a polynomial algebra with generators being irreducible and weight vectors.

Lemma

The hypersurfaces defined by the generators are just the orbital varieties closures lying in \mathfrak{m} .

Although this is almost a tautology, it is useful because an orbital variety closure is always given by an element w of the Weyl group, precisely as the B saturation set of $\mathfrak{n} \cap \mathfrak{n}^w$.

Benlolo and Sanderson suggested the form of these invariants in type A . It used my trick, motivated by quantum determinants, of using minors with 1 in the diagonal rather than 0.

The Benlolo-Sanderson conjecture was established in a paper with Melnikov (for Kirillov's 26th Birthday party).

It was shown that the resulting ideals could be strongly quantized. That is to say could be expressed as the graded ideal of the annihilator of a highest weight vector of a (simple) highest weight module, the interest being that then the annihilator of the module must be completely prime.

For \mathfrak{g} classical the resulting invariant generators were calculated by a student, Elena Perelman, but confirming to strict family tradition she did not publish the results despite being bribed with a prize that she accepted against family tradition.

She also showed that not all ideals could be strongly quantized, but they could all be weakly quantized, that is to say there are simple highest weight module whose associated variety is the given orbital variety closure.

Surprisingly the “minimal” orbit in type B admits orbital varieties closures which cannot even be weakly quantised.

16. Conclusion.

There are many easily stated problems concerning the invariant theory of finite dimensional Lie algebras which like the above are easy to formulate.

Many were inspired by the work of Bertram Kostant.

Many of these problems remain open.