Dirac Cohomology, Orbit Method and Unipotent Representations

Dedicated to Bert Kostant with great admiration

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Jordan decomposition

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unipotent representations $\rightarrow$ nilpotent orbits

Coadjoint orbits for $GL(n, \mathbb{R})$

$G = GL(n, \mathbb{R})$, its Lie algebra

$$\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R}) = \{\text{n by n real matrices}\}.$$

The adjoint action

$$Ad(g)(X) = gXg^{-1}.$$

The invariant trace from

$$\langle X, Y \rangle = \operatorname{tr} XY$$

defines an identification

$$\mathfrak{gl}(n, \mathbb{R})^* \longleftrightarrow \mathfrak{gl}(n, \mathbb{R}), \text{ and } f \mapsto X(f) \text{ is defined by}$$

$$f(Y) = \langle X(f), Y \rangle.$$

It sends coadjoint orbits to adjoint orbits.

Jordan decomposition

Def Suppose $X \in \mathfrak{gl}(n, \mathbb{R})$. We say that X is

nilpotent if $X^k = 0$ for some k;

semisimple if X is diagonalizable over $\mathbb{C}$;

elliptic if X is diagonalizable and all eigenvalues are imaginary;

hyperbolic if X is diagonalizable and all eigenvalues are real.

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Prop Given $X \in \mathfrak{gl}(n, \mathbb{R})$, there are unique X_h, X_e, X_n, s.t.

1) $X = X_h + X_e + X_n$.
2) X_h is hyperbolic, X_e is elliptic, and X_n is nilpotent.
3) X_h, X_e and X_n all commute with each other.
4) If Y commutes with X, then it commutes with X_h, X_e and X_n.

Real reductive groups

The *Cartan involution* for $GL(n,\mathbb{R})$ is the automorphism

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1) $\operatorname{Ker} \eta$ is finite;

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Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decompositions.

Coadjoint orbits for reductive groups

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Prop Suppose G is are real reductive group, and X is in $\mathfrak{g}_0$.

1) The Jordan components X_h, X_e, X_n are in $\mathfrak{g}_0$.

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Def (Jordan Decomposition)

Let $X(f) = X(f)_h + X(f)_e + X(f)_n$ be the Jordan decomposition.

Then the corresponding

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Orbit method for reductive groups (Vogan [ICM 1986])

Suppose that $f \in \mathfrak{g}_0^*$.

$G(f) =$ centralizer of $X(f)$ in G, $\mathfrak{g}_0(f) = \{Y \in \mathfrak{g}_0 \mid [X(f), Y] = 0\}$.

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$$\theta f_h = -f_h, \ \theta f_e = f_e, \text{ and}$$

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Since X_e and X_n commute with X_h, and so belong to $\mathfrak{g}(f_h)$, we can identify f_e and f_n (by restriction) with elements of $\mathfrak{g}(f_h)^*$.
Thus,

$$G(f_h) \supset [G(f_h)](f_e) \supset \{[G(f_h)](f_e)\}(f_n);$$

these are the same groups as

$$G(f_h) \supset G(f_s) \supset G(f).$$

$$\widehat{G(f)} \to \widehat{G(f_s)} \to \widehat{G(f_h)} \to \widehat{G}.$$

Hyperbolic step: parabolic induction

Fix $f_h \in \mathfrak{g}_0^*$ hyperbolic, let X_h be the corresponding element in $\mathfrak{g}_0$.

$$\text{We have } \mathfrak{g}_0 = \sum_{r \in \mathbb{R}} \mathfrak{g}_0^r, \ \mathfrak{g}_0^r = \{Y \in \mathfrak{g}_0 \mid [X_h, Y] = rY\}.$$

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Then $\mathfrak{g}_0^0 = \mathfrak{g}_0(f_h)$, $[\mathfrak{g}_0^r, \mathfrak{g}_0^s] \subset g_0^{r+s}$, and $\langle \mathfrak{g}_0^r, \mathfrak{g}_0^s \rangle = 0$ if $r + s \neq 0$.

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Set

$$\mathfrak{n}_h = \sum_{r>0} \mathfrak{g}_0^r, \text{ a nilpotent subalgebra normalized by } G(f_h).$$

Write $N_h = exp(\mathfrak{n}_h)$.

Then $P_h = G(f_h)N_h$ is a *parabolic subgroup* of G.

The hyperbolic step $\widehat{G(f_h)} \to \widehat{G}$ is defined by

$$\pi \mapsto \operatorname{Ind}_{P_h}^{G} \pi.$$

Elliptic step: cohomological induction

Fix $f_e \in \mathfrak{g}_0^*$ elliptic, write X_e for the corresponding element in $\mathfrak{g}_0$.

$$\text{We have } \mathfrak{g} = \sum_{r\in\mathbb{R}} \mathfrak{g}^r,\ \mathfrak{g}^r = \{Y \in \mathfrak{g} |\ [iX_e, Y] = rY\}.$$

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$$\mathfrak{u}_e = \sum_{r>0} \mathfrak{g}^r, \text{ and}$$

$$\mathfrak{q}_e = \mathfrak{g}(f_e) + \mathfrak{u}_e,$$

a θ-stable parabolic subalgebra normalized by $G(f_e)$.
The <u>elliptic step</u> $\widehat{G(f_e)} \to \widehat{G}$ is defined by

$$\pi \mapsto \mathcal{L}_{\mathfrak{q}_e} \pi.$$

Nilpotent step: unipotent representations (‘unipotents’)

Example The ‘unipotents’ attached to 0 must be trivial on G_0.

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It remains to define ‘unipotents’.

Dequantization

Given $\pi \in \widehat{G}$. Now we reverse the process of quantization and look for the corresponding $f = f_h + f_e + f_n \in \mathfrak{g}_0^*$ attached to π.

Question Which π corresponds a semsimple element $f = f_s = f_h + f_e$?

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The primary property of π is its infinitesimal character.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$.

Write $W = W(\mathfrak{g}, \mathfrak{h})$ for the Weyl group. Then we have

$$\text{Harish-Chandra isomorphsim } \xi \colon Z(\mathfrak{g}) \to S(\mathfrak{h})^W.$$

Infl char $\xi_\Lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$ is parametrized by W-orbits $W \cdot \Lambda$ in $\mathfrak{h}^*$.

Parabolic induction: reduction to real infl char

Let $H = TA$ be a θ-stable Cartan subgroup with CSA $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$. The canonical real form of the complexified CSA $\mathfrak{h}$ is

$$\operatorname{RE}\mathfrak{h} = i\mathfrak{t}_0 + \mathfrak{a}_0.$$

Then $\Lambda \in \mathfrak{h}^*$ has a unique decomposition

$$\Lambda = \operatorname{RE}\Lambda + i\operatorname{IM}\Lambda.$$

This decomposition is independent of choices of θ-stable Cartan subgroup H.

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Thm [Knapp] $\pi \in \widehat{G}$. There is a $P = MAN$ a parabolic subgroup and $\delta \in \widehat{M}$ with real infl char, and ν a unitary character of A, s.t.

$$\pi \cong \mathrm{Ind}_P^G(\delta \otimes \nu).$$

Cohomological induction in good range

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra.

Let L be the normalizer of $\mathfrak{q}$ in G.
We say that $Z \in \widehat{L}$ is in *good range*, if

$$\langle \Lambda_Z + \rho(\mathfrak{u}), \beta \rangle > 0, \forall \beta \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Thm (Vogan) The cohomological inducton $\mathcal{L}_{\mathfrak{q}}$ maps
$Z \in \widehat{L}$ in good range to a $\pi = \mathcal{L}_{\mathfrak{q}}(Z) \in \widehat{G}$.

Dirac operators and Vogan's conjecture

Let e_i be a basis for $\mathfrak{s}$ and f_i the dual basis with respect to the trace form. The Dirac operator is defined by

$$D \colon = \sum e_i \otimes f_i \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

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Vogan conjectured that D has the following property:

there is a natural algebra homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$, s.t. $\forall z \in Z(\mathfrak{g})$ one has

$$z \otimes 1 - \zeta(z) = Da + bD$$

for some a, b in $U(\mathfrak{g}) \otimes C(\mathfrak{s})$.

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The motivation of the conjecture is to show that

$$H_D(V) = H_D(V \otimes S):= \mathrm{Ker}\, D / \mathrm{Ker}\, D \cap \mathrm{Im}\, D$$

detects the infinitesimal character of V.

Dirac inequality and Dirac index

If π is unitary, then D is self-dual and $H_D(X_\pi) = \operatorname{Ker} D = \operatorname{Ker} D^2$.

If $\operatorname{Hom}_{\widetilde{K}}(E_\gamma, X_\pi \otimes S) \neq 0$, then

$$\langle \Lambda, \Lambda \rangle \leq \langle \gamma + \rho_c, \gamma + \rho_c \rangle.$$

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Thm [H-Pandzic] The followings are equivalent:

1) $H_D(X_\pi) \neq 0$;
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If dim G/K is even, then $S = S^+ \oplus S^-$ and the index
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Vogan's conjecture tells how to calculate this index.

Remark Vogan's conjetcure has been extended to many different setting: by Kostant to general $(\mathfrak{g}, \mathfrak{r})$, by Frajria-Kac-Papi to affine Lie algebras, by Kumar for equivariant cohomology, by Barbasch-Cibutaru-Trapa to graded Hecke algebras, ...

Parabolic induction: further reduction to $H_D(X) \neq 0$

Conjecture:
Suppose that $\pi \in \widehat{G}$ with real infl char and $H_D(X_\pi) = 0$.

Then there is

1) a $P = MAN$ a parabolic subgroup,
2) $\sigma \in \widehat{M}$ with $H_D(X_\sigma) \neq 0$,
3) ν a non-unitary character of A or trivial, s.t.
$\mathrm{Ind}_P^G(\delta \otimes t\nu)(0 \leq t \leq 1)$ is unitary and irreducible, and

$$\pi \cong \mathrm{Ind}_P^G(\delta \otimes \nu).$$

Linear groups, regular infl char

Suppose that G_{ell} the set of regular elliptic elements is not empty and open.

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Harish-Chandra showed that global characters of discrete series are completely determined by the restriction to G_{ell}.

Linear groups, regular infl char

Suppose that G_{ell} the set of regular elliptic elements is not empty and open.

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If G is linear and the infi char Λ of $\pi \in \widehat{G}$ is regular, then Λ is strongly regular and

$$X_\pi \cong A_{\mathfrak{q}}(\lambda)$$

is cohomiologically induced from 1-dim representation [Salamanca].

$$H_D(A_{\mathfrak{q}}(\lambda)) = \bigoplus_{w \in W_{\mathfrak{l}}^1} E_{w(\lambda+\rho)}.$$

H-Pandzic-Vogan [SciChinaMath 2017] showed that

$$H_D(X_\pi) \text{ essentially determines } X_\pi = A_{\mathfrak{q}}(\lambda).$$

Definition of unipotents for linear groups

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Def Suppose that G is linear and $\pi \in \widehat{G}$ has infl char Λ.
We say that π is *unipotent*, if

1) $H_D(X_\pi) = \bigoplus_{w \in W^1} E_{w\Lambda}$,

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Remark If π is unipotent and not trivial, then

1) infl char of π is singular,

2) infl char of π is small (inside the convex hull of $W \cdot \rho$).

Unipotents: $A_{\mathfrak{q}}(\lambda)$ at weakly fair edge

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra.

Let L be the normalizer of $\mathfrak{q}$ in G.

We say that λ or $A_{\mathfrak{q}}(\lambda)$ is at *weakly fair edge*, if

$$\langle \lambda|_{\mathfrak{c}}, \beta \rangle = 0, \ \forall \beta \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Here $\mathfrak{c}$ is the center of $\mathfrak{l}$.

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Example $SL(2n, \mathbb{R})$: Speh representations at the weakly fair edge

Infl char $= (n-1)e_1 + (n-3)e_2 + \cdots + (-n+1)e_n$;
LKT $= (1, \cdots, 1, \pm 1)$.

Both are $A_{\mathfrak{q}}(\lambda)$-module at weakly fair edge with two different θ-stable $\mathfrak{q}$ (pp. 586-588 [Knapp-Vogan]).

Question Is every unipotent an $A_{\mathfrak{q}}(\lambda)$ at weakly fair edge?

Definition of unipotents for nonlinear groups

Def Suppose that G is reductive and $\pi \in \widehat{G}$ has infl char Λ. We say that π is called *unipotent* if

$$1) \quad H_D(X_\pi) = \bigoplus_{w \in W^1_{[\Lambda]}} E_{w\Lambda},$$

where Λ is the infl char of a $\widetilde{K}$-type in $H_D(X_\pi)$ and $W_{[\Lambda]}$ is the integral Weyl group.

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Remark Some unipotents show up in pairs (as twins).

If π and π' are attached to the same orbit and satisfy

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Example The two irreducible components of the oscillator representation are associate.

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Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra.

Let L be the normalizer of $\mathfrak{q}$ in G.

We say that $A_{\mathfrak{q}}(\lambda)$ is near weakly fair edge if

$$|\langle \lambda|_{\mathfrak{c}}, \beta^{\vee} \rangle| < 1, \forall \beta \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

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Fact If λ is at weakly fair edge and weakly unipotent, then $A_{\mathfrak{q}}(\lambda)$ is unitary. (Vogan [Invent 1985] pp 492-493.)

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Example Speh representations for $\widetilde{SL(2n, \mathbb{R})}$, infl char $\frac{1}{2}\rho$.

There are four (two pairs of associates): Lucas [TAMS, 2008].

Their LKT are $\frac{1}{2}(1, \cdots, 1, \pm 1)$ and $\frac{3}{2}(1, \cdots, 1, \pm 1)$.

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Remark There is one pair of associates for $\widetilde{GL(2n, \mathbb{R})}$.

Unipotents: the Oscillator representation

$G = Sp(2n, \mathbb{R})$.

Let π and π' be the two irreducible components of the oscillator representation.

Both are attached to a minimal orbit.

Their infl char Λ is $\rho(B_n)$ which is regular, and $W_{[\Lambda]}$ is of type D_n (same as the Weyl group for $SO^*(2n)$).

$$H_D(X_\pi) \oplus H_D(X_{\pi'}) = \bigoplus_{w \in W^1} E_{w\Lambda}.$$

Neither π nor π' is an $A_{\mathfrak{q}}(\lambda)$-module.

They are theta lifts of trivial and sign representations of $O(1)$.

Unipotents in highest weight modules: Wallach modules

$G = U(p,q), O^*(2n)$:
H-Pandzic-Protsak [PJM 2011] showed that Dirac cohomology of any Wallach representation π

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$G = Sp(2n, \mathbb{R})$: any Wallach representation π has an associate π'.

$$H_D(X_\pi) \oplus H_D(X_{\pi'}) = \bigoplus_{w \in W^1} E_{w\Lambda}.$$

They are theta lift from trivial and the sign representations of $O(k)$ $(k = 1, \ldots, n)$.

Unipotents and corresponding nilpotent orbits

The unipotents are constructed from

1) $A_{\mathfrak{q}}(\lambda)$ at (or near) the weakly fair edge

2) Arthur's packets

3) Theta lifting

4) Unitary degenerate principal series:

    Rothschild-Wolf [Annales of E.N.S. 1974]

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Problem Determine all unipotents and their correponding nilpotent orbits (data).

THANK YOU!